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OPTIMALITY CONDITIONS AND SENSITIVITY ANALYSIS IN SOME CLASSES OF OPTIMIZATION PROBLEMS

Speciality: Applied Mathematics
Speciality code: 9 46 01 12

SUMMARY OF DOCTORAL THESIS

Thai Nguyen – 2026

The thesis was written on the basis of the author's research works carried at Thai Nguyen University of Sciences.

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First referee: .....
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Second referee: .....
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Third referee: .....
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To be defended at the Council of Thai Nguyen University:
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on ....., at .....o'clock .....

The thesis is publicly available at:

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- .....
- .....

List of published works related to the thesis

(CT1) D.T.V. An, D.T. Ngoan, N.V. Tuyen, "Optimality conditions and sensitivity analysis in parametric convex minimax programming", *Applicable Analysis* 103, 2997–3016 (2024)

(CT2) D.T.V. An, N.H. Hung, D.T. Ngoan, N.V. Tuyen, "Optimality conditions and sensitivity analysis in parametric nonconvex minimax programming", *Journal of Global Optimization* 90, 53–72 (2024)

(CT3) D.T. Ngoan, C. Gutiérrez, D.T.V. An, "Two-variable domination structures and applications in vector optimization", *Journal of Optimization Theory and Applications* 208, 36 (2026)

Introduction

Optimality conditions and sensitivity analysis are two central topics in optimization theory. When an optimization problem involves parameters, the optimal value function behaves as a function of the parameters. Notably, the optimal value function may be non-differentiable with respect to the parameters, even when all initial data of the problem are differentiable. Consequently, studies on sensitivity analysis are usually conducted through generalized derivatives or subdifferentials. To establish optimality conditions for problems with non-smooth data, generalized derivative concepts are employed. These concepts are constructed based on two main approaches: the primal approach on the underlying space and the dual approach on the dual space. Both approaches are effective, but in certain cases, the dual approach proves to be superior. The differentiation theory for the class of non-smooth functions, along with its important applications in mathematical programming and optimal control, was developed by many mathematicians, most notably the theory established by Mordukhovich in the 1980s. This theory continues to be developed to this day, yielding new advancements.

It is important to emphasize that to obtain the calculus rules of non-smooth analysis, appropriate qualification conditions are required. Relaxing these conditions helps extend the scope of applicability of the research results. The metric qualification condition, formulated via distance functions, was first studied and developed in the works of Ioffe, Penot, Jourani, and Thibault, and more recently by Huong et al. for the nonconvex case. According to Ioffe, the metric qualification condition is even weaker than the weakest sufficient conditions used in convex analysis. The relationship between the metric qualification condition and other sufficient conditions has been investigated, thereby allowing a more comprehensive exploration of optimality conditions and sensitivity analysis for parametric optimization problems.

A minimax optimization problem belongs to a class of optimization problems that considers both minimization and maximization processes simultaneously. These problems find numerous applications in mathematics, economics, and computer

science, such as the best approximation polynomial in the Chebyshev sense, multi-objective optimization, and game theory. According to Demyanov and Malozemov, minimax optimization problems can be divided into two categories: discrete minimax problems and continuous minimax problems. The best approximation in the Chebyshev sense can be modeled as a discrete minimax problem, whereas matrix games lead to a continuous minimax problem. An essential part of the topics in minimax optimization problems is the study of optimality conditions as well as duality results. Authors have utilized the Mordukhovich subdifferential and several other advanced tools from variational analysis. Among these, Lipschitz continuity is a fundamental factor to derive the subdifferential calculus rule for the sum of functions, as well as the subdifferential rule for the maximum function. Regarding the topic of sensitivity analysis in parametric minimax optimization problems, to the best of our knowledge, the results by Darkhovskii and Levitin in 1976 were the earliest. Specifically, the authors obtained a differential expansion of the optimal value function in the neighborhood of a given point (Theorems 1 and 2). To date, we have not found any further works investigating sensitivity analysis for parametric minimax optimization problems. Therefore, this dissertation focuses on establishing optimality conditions and sensitivity analysis for parametric minimax optimization problems in both convex and nonconvex cases, based on two types of qualification conditions: the metric qualification condition and the continuity (or Lipschitz continuity) condition of the component functions.

The second class of problems we investigate is the vector optimization problem. It should be emphasized that in vector optimization, the concept of an optimal solution is one of the core elements. This concept is typically constructed under the assumption that the image space Y of the problem is partially ordered by a nontrivial pointed convex cone. In the finite-dimensional case, the most common ordering cone is the non-negative orthant cone, consisting of points whose coordinates are all non-negative. This corresponds exactly to the multi-objective optimization problem frequently encountered in practice. The partial order relation generated by the non-negative orthant cone leads to the concept of an optimal solution in the Pareto sense. It is worth noting that modeling comparison relations in this manner, i.e., using a fixed ordering cone, does not yet encompass all scenarios arising in reality. This limitation has been recognized in studies on decision-making processes, multi-objective optimization, and medical image registration.

To improve upon this, researchers have used variable domination structures to replace the fixed domination structure. Accordingly, each element of the image

space Y has its own ordering cone. This cone is determined by a cone-valued set-valued mapping, usually denoted as $\mathcal{D} : Y \rightrightarrows Y$. The idea of variable domination structures was first introduced by Yu in 1974. Subsequently, Eichfelder et al. built a general and relatively comprehensive theoretical framework for variable domination structures, as well as their applications in vector optimization, including: optimality conditions for non-smooth problems using subdifferentials and coderivatives, linear and nonlinear scalarization results, duality, and numerical methods.

Up to now, although existing under various names, there are generally two fundamental concepts of optimal points related to variable domination structures: one concept is based on the condition that the point under consideration is nondominated by any other point according to the ordering cone of that point itself; the other concept is based on the condition that the point under consideration is nondominated by any other point according to the ordering cones of those other points (not the point under consideration). We observe that in the existing literature, based on these two optimization concepts, all related properties are also formulated in two different but closely connected ways. Therefore, this dissertation aims to introduce a more general domination structure that encompasses both of the aforementioned concepts.

The objective of the dissertation is to study and establish first-order necessary and sufficient optimality conditions and sensitivity analysis for parametric discrete minimax optimization problems; to define and study the properties of two-variable domination structures in linear spaces, and to establish optimality conditions for vector optimization problems with two-variable domination structures via scalarization methods.

The content of the dissertation is divided into three chapters, along with the introduction, conclusion, list of notations, list of the author's scientific works, and references.

Chapter 1. Preliminaries

Chapter 2. Optimality conditions and sensitivity analysis for parametric minimax optimization problems

Chapter 3. Solution characterizations for vector optimization problems with two-variable domination structures

Chapter 1

Preliminaries

This chapter presents several relevant concepts and results in variational analysis, convex analysis, and optimization theory. Among these, the core contents include: order relations in linear spaces, optimal solution concepts for vector optimization problems; the normal cone of a set, the subdifferential of a single-valued function, the coderivative of a set-valued mapping, and the metric qualification condition.

1.1 Vector optimization problems and related concepts

1.1.1 Order relation in linear spaces

1.1.2 Vector optimization problems

1.2 Minimax optimization problems

1.2.1 Examples of minimax optimization problems

1.2.2 Formulation of parametric discrete minimax problems

Let X and Y be two Banach spaces and m be a given positive integer. Consider the parametric discrete minimax optimization problem

$$\min_{y\in G(x)} \max_{k\in J} \varphi_k(x,y), \tag{P_x}$$

where x is the parameter, $G\colon X \rightrightarrows Y$ is a set-valued mapping, $J := \{1,\ldots,m\}$, and the functions $\varphi_k\colon X\times Y \to \overline{\mathbb{R}}$, $k\in J$. In the following sections, we denote the maximum function as $\varphi(x,y) := \max_{k\in J}\varphi_k(x,y)$ for all $(x,y)\in X\times Y$. Then, the optimal value function $\mu\colon X\to\overline{\mathbb{R}}$ of problem (P_x) is

$$\mu(x) = \inf_{y\in G(x)} \varphi(x,y),$$

with the convention that $\mu(x)$ equals $+\infty$ if $x \notin \operatorname{dom} G$. We define the solution mapping $M\colon \operatorname{dom} G \rightrightarrows Y$ of (P_x) as $M(x) := \{y \in G(x) \,:\, \mu(x) = \varphi(x,y)\}$.

1.3 Other auxiliary concepts and results

1.3.1 Normal cone

1.3.2 Subdifferential

1.3.3 Coderivative

1.3.4 Metric qualification condition

The metric qualification condition is formulated via distance functions as follows.

Definition 1.40. Let $C_1, ..., C_n$ be nonempty subsets of X. We say that the sets $C_i, i = 1, ..., n$, satisfy the *metric qualification condition* (MQC) at the point $\bar{x} \in C := \cap_{i=1}^n C_i$ if there exist two positive real numbers $\gamma > 0$ and $\delta > 0$ such that

$$d(x, C) \leq \gamma[d(x, C_1) + \cdots + d(x, C_n)], \quad \forall x \in \mathbb{B}(\bar{x}, \delta). \tag{MQC}$$

Chapter 2

Optimality conditions and sensitivity analysis for parametric minimax optimization problems

This chapter presents the main results we have investigated for parametric minimax problems. These results include: the subdifferential calculus rule for the maximum function under the metric qualification condition (the corresponding result under the continuity condition was stated in Chapter 1); first-order necessary and sufficient optimality conditions; and sensitivity analysis for parametric minimax problems in both convex and nonconvex cases.

2.1 Subdifferential rule for the maximum function

Consider the maximum function $f(x) := \max\{f_i(x) : i = 1, \ldots, m\}$ defined on an Asplund space X, where $f_i : X \to \overline{\mathbb{R}}, i = 1, ..., m$. Let $I(\bar{x})$ denote the active index set at $\bar{x}$, and

$$\Lambda(\bar{x}) := \left\{(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ : \sum_{i=1}^m \lambda_i = 1, \lambda_i[f_i(\bar{x}) - f(\bar{x})] = 0, i = 1, \ldots, m\right\},$$

$$\bar{\Lambda}(\bar{x}) := \left\{(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m : \sum_{i=1}^m \lambda_i = 1, \lambda_i[f_i(\bar{x}) - f(\bar{x})] = 0, i = 1, \ldots, m\right\}.$$

First, we consider the case where all functions f_i, $i = 1, \ldots, m$, are convex. Then f is also a convex function and we have the following result.

Theorem 2.1. *For every $\bar{x} \in \bigcap_{i=1}^m \operatorname{dom} f_i$, we always have*

$$\partial f(\bar{x}) \supset \bigcup \left\{\sum_{i \in I(\bar{x})} \lambda_i \circ \partial f_i(\bar{x}),\ (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x})\right\},$$

where $\lambda_i \circ \partial f_i(\bar{x})$ is defined as $\lambda_i \partial f_i(\bar{x})$ if $\lambda_i > 0$ and as $\partial^\infty f_i(\bar{x})$ if $\lambda_i = 0$. In addition, if $\operatorname{epi} f_i$, $i \in I(\bar{x})$, satisfies the metric qualification condition (MQC) *at*

$(\bar{x}, f(\bar{x}))$ and if f_i, $i \notin I(\bar{x})$, is continuous at $\bar{x}$, then we have

$$\partial f(\bar{x}) = \bigcup \left\{ \sum_{i \in I(\bar{x})} \lambda_i \circ \partial f_i(\bar{x}),\ (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}) \right\},$$

and

$$\partial^\infty f(\bar{x}) = \sum_{i \in I(\bar{x})} \partial^\infty f_i(\bar{x}).$$

The next theorem presents the evaluation result for the Mordukhovich subdifferential of the maximum function under the metric qualification condition.

Theorem 2.6. *Let $\bar{x} \in X$. Assume that f_i is l.s.c. for $i \in I(\bar{x})$ and is u.s.c. at $\bar{x}$ for $i \notin I(\bar{x})$. Furthermore, assume that there exist $\gamma > 0$ and $\delta > 0$ such that*

$$d((x, r), \operatorname{epi} \bar{f}) \leq \gamma \sum_{i \in I(\bar{x})} d((x, r), \operatorname{epi} f_i),\ \forall x \in \mathbb{B}(\bar{x}, \delta), r \in \mathbb{B}(f(\bar{x}), \delta),$$

where $\bar{f} := \max_{i \in I(\bar{x})} f_i$. Then, we have

$$\partial_L f(\bar{x}) \subset \bigcup \left\{ \sum_{i \in I(\bar{x})} \lambda_i \circ \partial_L f_i(\bar{x}),\ (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}) \right\},$$

$$\partial_L^\infty f(\bar{x}) \subset \sum_{i \in I(\bar{x})} \partial_L^\infty f_i(\bar{x}),$$

where $\lambda \circ \partial_L f(\bar{x})$ is defined as $\lambda \partial_L f(\bar{x})$ when $\lambda > 0$ and as $\partial_L^\infty f(\bar{x})$ when $\lambda = 0$.

2.2 Optimality conditions and sensitivity analysis for parametric convex minimax problems

In this section, we consider the parametric discrete minimax optimization problem (P_x) under the assumption that the functions $\varphi_k, k \in J$, are proper convex functions. The main results include optimality conditions and sensitivity analysis for parametric convex minimax optimization problems. The first part presents theorems on necessary and sufficient optimality conditions using two types of qualification conditions: the metric qualification condition and the continuity condition. The second part is dedicated to sensitivity analysis with calculus results for the subdifferential of the optimal value function.

2.2.1 Necessary and sufficient optimality conditions

Optimality conditions for convex optimization problems are derived from the calculus rules of convex analysis, which have been presented in many monographs

and scientific papers. In this part, we consider convex minimax problems where the objective function φ is the maximum of a finite number of convex functions.

Theorem 2.7 (Necessary and sufficient optimality conditions I)**.** *Consider problem $(P_{\bar{x}})$ with $\bar{x} \in X$ and $\bar{y} \in \bigcap_{k\in J} \operatorname{dom} \varphi_k(\bar{x}, \cdot)$. Assume that $\varphi_k(\bar{x}, .)$, $k \in J$, are convex functions and $G(\bar{x})$ is a convex set. Furthermore, assume that*

(i) $\operatorname{epi} \varphi(\bar{x}, \cdot)$ *and* $G(\bar{x}) \times \mathbb{R}$ *satisfy condition* (MQC) *at* $(\bar{y}, \varphi(\bar{x}, \bar{y}))$,

(ii) $\operatorname{epi} \varphi_k(\bar{x}, \cdot)$, $k \in I(\bar{x}, \bar{y})$, *satisfy condition* (MQC) *at* $(\bar{y}, \varphi(\bar{x}, \bar{y}))$ *and* $\varphi_k(\bar{x}, \cdot)$, $k \notin I(\bar{x}, \bar{y})$, *are continuous at* $\bar{y}$.

Then, $\bar{y}$ is a global solution of problem $(P_{\bar{x}})$ if and only if

$$0 \in \bigcup \left\{ \sum_{k\in I(\bar{x},\bar{y})} \lambda_k \circ \partial_y \varphi_k(\bar{x}, \bar{y}), (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}, \bar{y}) \right\} + N(\bar{y}; G(\bar{x})),$$

where $I(\bar{x}, \bar{y}) := \{k \in J : \ \varphi_k(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y})\}$,

$$\Lambda(\bar{x}, \bar{y}) := \left\{ (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ : \sum_{k\in J} \lambda_k = 1, \lambda_k[\varphi_k(\bar{x}, \bar{y}) - \varphi(\bar{x}, \bar{y})] = 0, k \in J \right\}$$

and $\lambda_k \circ \partial_y \varphi_k(\bar{x}, \bar{y})$ is defined as $\lambda_k \partial_y \varphi_k(\bar{x}, \bar{y})$ if $\lambda_k > 0$ and as $\partial^\infty_y \varphi_k(\bar{x}, \bar{y})$ if $\lambda_k = 0$.

Using the continuity condition, we obtain the necessary and sufficient optimality conditions for problem (P_x) as follows.

Theorem 2.8 (Necessary and sufficient optimality conditions II)**.** *Assume that $\bar{x} \in X$ and $\bar{y} \in \bigcap_{k\in J} \operatorname{dom} \varphi_k(\bar{x}, \cdot)$. Suppose that $\varphi_k(\bar{x}, .)$, $k \in J$, are convex functions and $G(\bar{x})$ is a convex set. Furthermore, assume that all $\varphi_k(\bar{x}, \cdot)$, $k \in J$, are continuous at $\bar{y}$. Then, $\bar{y}$ is a global solution of problem (P_x) if and only if*

$$0 \in \bigcup \left\{ \sum_{k\in I(\bar{x},\bar{y})} \lambda_k \partial_y \varphi_k(\bar{x}, \bar{y}), (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}, \bar{y}) \right\} + N(\bar{y}; G(\bar{x})).$$

2.2.2 Sensitivity analysis

In this part, we will establish formulas for evaluating/computing the subdifferential and singular subdifferential of the optimal value function μ defined by (1.2.2) via the normal cone of $\operatorname{gph} G$ and the subdifferentials of $\varphi_k, k \in J$.

Theorem 2.11. *Consider problem (P_x). Assume that the mapping G has a convex graph, the functions φ_k, $k \in J$, are convex on $X \times Y$, and the optimal value function μ is finite at $\bar{x} \in \operatorname{dom} M$. In addition, assume that $\bar{y} \in M(\bar{x})$ and*

(i) $\operatorname{epi}\varphi$ *and* $\operatorname{gph} G \times \mathbb{R}$ *satisfy condition* (MQC) *at* $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$*;*

(ii) $\operatorname{epi}\varphi_k, k \in I(\bar{x}, \bar{y})$*, satisfy condition* (MQC) *at* $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$ *and the functions* $\varphi_k, k \notin I(\bar{x}, \bar{y})$*, are continuous at* $(\bar{x}, \bar{y})$*.*

Then, $x^* \in \partial\mu(\bar{x})$ *if and only if*

$$(x^*, 0) \in \bigcup \left\{ \sum_{k \in I(\bar{x}, \bar{y})} \lambda_k \circ \partial\varphi_k(\bar{x}, \bar{y}), (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}, \bar{y}) \right\} + N((\bar{x}, \bar{y}); \operatorname{gph} G),$$

where $\lambda_k \circ \partial\varphi_k(\bar{x}, \bar{y})$ *is defined as* $\lambda_k \partial\varphi_k(\bar{x}, \bar{y})$ *if* $\lambda_k > 0$ *and as* $\partial^\infty \varphi_k(\bar{x}, \bar{y})$ *if* $\lambda_k = 0$*.*

The singular subdifferential of μ is computed by the following formula.

Theorem 2.12. *Assume that the mapping* G *has a convex graph, the functions* φ_k*,* $k \in J$*, are convex on* $X \times Y$*, and the optimal value function* μ *is finite at* $\bar{x} \in \operatorname{dom} M$*. In addition, assume that* $\bar{y} \in M(\bar{x})$ *and*

(i) $\operatorname{dom}\varphi \times \mathbb{R}_+$ *and* $\operatorname{gph} G \times \mathbb{R}$ *satisfy condition* (MQC) *at* $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$*;*

(ii) $\operatorname{epi}\varphi_k, k \in I(\bar{x}, \bar{y})$*, satisfy condition* (MQC) *at* $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$ *and the functions* $\varphi_k, k \notin I(\bar{x}, \bar{y})$*, are continuous at* $(\bar{x}, \bar{y})$*.*

Then, we have

$$\partial^\infty \mu(\bar{x}) = \left\{ x^* \in X^* : (x^*, 0) \in \sum_{k \in I(\bar{x}, \bar{y})} \partial^\infty \varphi_k(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}); \operatorname{gph} G) \right\}.$$

Under the continuity assumption of the functions φ_k, we have the following result.

Theorem 2.13. *Assume that the mapping* G *has a convex graph, the functions* φ_k*,* $k \in J$*, are convex on* $X \times Y$*, and the optimal value function* μ *is finite at* $\bar{x} \in \operatorname{dom} M$*. Furthermore, assume that* φ_k *is continuous at* $(\bar{x}, \bar{y})$*. Then,* $x^* \in \partial\mu(\bar{x})$ *if and only if*

$$(x^*, 0) \in \bigcup \left\{ \sum_{k \in I(\bar{x}, \bar{y})} \lambda_k \partial\varphi_k(\bar{x}, \bar{y}), (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}, \bar{y}) \right\} + N((\bar{x}, \bar{y}); \operatorname{gph} G),$$

and

$$\partial^\infty \mu(\bar{x}) = \left\{ x^* \in X^* : (x^*, 0) \in \sum_{k \in I(\bar{x}, \bar{y})} \partial^\infty \varphi_k(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}); \operatorname{gph} G) \right\}.$$

2.3 Optimality conditions and sensitivity analysis for parametric nonconvex minimax problems

Consider the parametric discrete minimax optimization problem (P_x) under the assumption that X, Y are Asplund spaces and the functions $\varphi_k, k \in J$, are proper

functions.

2.3.1 Necessary and sufficient optimality conditions

First, we present the necessary optimality conditions using the metric qualification condition for local optimal solutions of problem (P_x).

Theorem 2.15 (Necessary optimality conditions I)**.** *Assume that $\bar{x} \in X$, $\varphi_k(\bar{x}, .)$, $k \in J$, are l.s.c., and $G(\bar{x})$ is a closed set. Furthermore, assume that $\bar{y} \in Y$ is a local optimal solution of problem $(P_{\bar{x}})$ and*

(i) $\operatorname{epi} \bar{\varphi}(\bar{x}, \cdot)$ *and* $G(\bar{x}) \times \mathbb{R}$ *satisfy condition* (MQC) *at* $(\bar{y}, \varphi(\bar{x}, \bar{y}))$*, where* $\bar{\varphi}(x, y) := \max\limits_{k \in I(\bar{x}, \bar{y})} \varphi_k(x, y)$*;*

(ii) $\varphi_k(\bar{x}, \cdot), k \notin I(\bar{x}, \bar{y})$*, is u.s.c. at* $\bar{y}$ *and the sets* $\operatorname{epi} \varphi_k(\bar{x}, \cdot)$, $k \in I(\bar{x}, \bar{y})$*, satisfy condition* (MQC) *at* $(\bar{y}, \varphi(\bar{x}, \bar{y}))$.

Then, we have

$$0 \in \bigcup \left\{ \sum_{k \in I(\bar{x}, \bar{y})} \lambda_k \circ \partial_{L,y} \varphi_k(\bar{x}, \bar{y}), (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}, \bar{y}) \right\} + N_L(\bar{y}; G(\bar{x})), \quad (2.28)$$

where $\lambda_k \circ \partial_{L,y} \varphi_k(\bar{x}, \bar{y})$ is defined as $\lambda_k \partial_{L,y} \varphi_k(\bar{x}, \bar{y})$ if $\lambda_k > 0$ and as $\partial^{\infty}_{L,y} \varphi_k(\bar{x}, \bar{y})$ if $\lambda_k = 0$.

The following theorem presents the necessary condition for local optimal solutions of problem (P_x) using the Lipschitz continuity condition.

Theorem 2.16 (Necessary optimality conditions II)**.** *Assume that $\bar{x} \in X$, $G(\bar{x})$ is a closed set, and $\varphi_k(\bar{x}, \cdot)$, $k \in J$, are Lipschitz continuous functions around $\bar{y} \in Y$. If $\bar{y}$ is a local optimal solution of (P_x), then*

$$0 \in \bigcup \left\{ \sum_{k \in I(\bar{x}, \bar{y})} \lambda_k \partial_{L,y} \varphi_k(\bar{x}, \bar{y}), (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}, \bar{y}) \right\} + N_L(\bar{y}; G(\bar{x})).$$

The next theorem establishes sufficient conditions for optimal solutions of problem (P_x) using convexity.

Theorem 2.18 (Sufficient optimality conditions)**.** *Assume that $\bar{x} \in X$, $\bar{y} \in G(\bar{x})$, and the problem satisfies condition* (2.28)*. If $\varphi_k(\bar{x}, \cdot)$, $k \in J$, are convex functions and $G(\bar{x})$ is a convex set, then $\bar{y}$ is a global solution of $(P_{\bar{x}})$, i.e., $\bar{y} \in M(\bar{x})$.*

2.3.2 Sensitivity analysis

In this part, we present an upper estimate for the Mordukhovich subdifferential of the optimal value function μ of problem (P_x). These estimates are established through the Mordukhovich subdifferentials of $\varphi_k, k \in J$, and the Mordukhovich normal cone of gph G.

First, recall that the solution mapping $M : X \rightrightarrows Y$ of the parametric optimization problem (P_x) is called μ-inner semicontinuous at $(\bar{x}, \bar{y}) \in \operatorname{gph} M$ if for each sequence $x_k \xrightarrow{\mu} \bar{x}$, there exists a sequence $y_k \in M(x_k)$ converging to $\bar{y}$ as $k \to +\infty$.

Theorem 2.23. *Consider problem (P_x). Assume that G has a closed graph, φ_k, $k \in J$, are l.s.c., the optimal value function μ takes finite values at $\bar{x} \in \operatorname{dom} M$, and $\bar{y} \in M(\bar{x})$. Furthermore, assume that*

(i) *The solution mapping M is μ-inner semicontinuous at $(\bar{x}, \bar{y}) \in \operatorname{gph} M$;*

(ii) $\operatorname{epi} \bar{\varphi}$ *and* $\operatorname{gph} G \times \mathbb{R}$ *satisfy condition* (MQC) *at* $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$, *where* $\bar{\varphi} := \max_{k \in I(\bar{x}, \bar{y})} \varphi_k$;

(iii) $\varphi_k(\bar{x}, \bar{y})$ *is l.s.c. around* $(\bar{x}, \bar{y})$ *for* $k \in I(\bar{x}, \bar{y})$ *and u.s.c. at* $(\bar{x}, \bar{y}), k \notin I(\bar{x}, \bar{y})$, *and* $\operatorname{epi} \varphi_k$ *satisfies condition* (MQC) *at* $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$.

Then, if $x^* \in \partial_L \mu(\bar{x})$, we have

$$(x^*, 0) \in \bigcup \left\{ \sum_{k \in I(\bar{x}, \bar{y})} \lambda_k \circ \partial_L \varphi_k(\bar{x}, \bar{y}), (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}, \bar{y}) \right\} + N_L((\bar{x}, \bar{y}); \operatorname{gph} G),$$

where $\lambda_k \circ \partial_L \varphi_k(\bar{x}, \bar{y})$ is defined as $\lambda_k \partial_L \varphi_k(\bar{x}, \bar{y})$ if $\lambda_k > 0$ and as $\partial_L^\infty \varphi_k(\bar{x}, \bar{y})$ if $\lambda_k = 0$.

Under the assumption of Lipschitz continuity of the functions φ_k, we have the following result.

Theorem 2.24. *Consider problem (P_x). Assume that the optimal value function μ takes finite values at $\bar{x} \in \operatorname{dom} M$ and $\bar{y} \in M(\bar{x})$. Furthermore, assume that G has a closed graph, φ_k are Lipschitz continuous around $(\bar{x}, \bar{y})$ for $k \in J$, and the solution mapping M is μ-inner semicontinuous at $(\bar{x}, \bar{y}) \in \operatorname{gph} M$. Then, if $x^* \in \partial_L \mu(\bar{x})$, we have*

$$(x^*, 0) \in \bigcup \left\{ \sum_{k \in I(\bar{x}, \bar{y})} \lambda_k \partial_L \varphi_k(\bar{x}, \bar{y}), (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}, \bar{y}) \right\} + N_L((\bar{x}, \bar{y}); \operatorname{gph} G).$$

Chapter 3

Solution characterizations for vector optimization problems with two-variable domination structures

The first part of this chapter presents the definition of a two-variable domination structure. This concept extends the conventional variable domination structure concept. The subsequent part investigates related properties, defines the optimal solution concepts based on the two-variable domination structure, and compares them with existing definitions and properties based on conventional variable domination structures. The final part of the chapter presents approaches using nonlinear scalarization functions to obtain optimality conditions for local solutions of vector optimization problems with two-variable domination structures.

3.1 Two-variable domination structures

3.1.1 Definitions and properties

In this section, we define a two-variable domination structure on a real topological linear space Y and study its related properties. Specifically, consider a set-valued mapping $C : Y \times Y \rightrightarrows Y$ satisfying $\operatorname{dom} C = Y \times Y$, that is, $C(y_1, y_2)$ is nonempty for all $(y_1, y_2) \in Y \times Y$.

Definition 3.1. Let $y_1, y_2 \in Y$. We define

$$y_1 \leq^C y_2 :\Leftrightarrow y_2 \in y_1 + C(y_1, y_2).$$

The set-valued mapping C is called a two-variable ordering mapping and $\leq^C$ is called a two-variable domination structure on Y.

Next, to study the properties of the two-variable ordering mapping C, we define two set-valued mappings $D_C, I_C : Y \rightrightarrows Y$ as follows:

$$D_C(y) := \{d \in Y : y \leq^C y + d\},$$

$$I_C(y) := \{d \in Y : y - d \leq^C y\}.$$

The next lemma presents some properties related to the mappings D_C, I_C.

Lemma 3.3. *The following properties hold:*

(i) *For all $y, z \in Y$,*

$$y \leq^C z \Leftrightarrow z \in y + D_C(y) \Leftrightarrow y \in z - I_C(z).$$

(ii) *$D_C(y)$ and $I_C(y)$ can be represented in the form*

$$D_C(y) = \bigcup_{d \in Y} [\{d\} \cap C(y, y + d)]$$

and

$$I_C(y) = \bigcup_{d \in Y} [\{d\} \cap C(y - d, y)].$$

In addition, $d \in D_C(y) \Leftrightarrow d \in I_C(y + d)$, $d \in I_C(y) \Leftrightarrow d \in D_C(y - d)$.

(iii) *Consider a set-valued mapping $\mathcal{D} : Y \rightrightarrows Y$. Then $D_{C_1^{\mathcal{D}}} = \mathcal{D}$ (correspondingly, $I_{C_2^{\mathcal{D}}} = \mathcal{D}$).*

Next, we state some properties of the two-variable domination structure.

Theorem 3.5. *Let $C : Y \times Y \rightrightarrows Y$ be a two-variable ordering mapping.*

(i) *The binary relation $\leq^C$ is reflexive if and only if $0 \in C(y, y)$, for all $y \in Y$.*

(ii) *The binary relation $\leq^C$ is transitive if and only if*

$$\bigcup_{z \in y + D_C(y)} (z + D_C(z)) \subset y + D_C(y), \quad \forall y \in Y$$

or

$$\bigcup_{z \in w - I_C(w)} (z - I_C(z)) \subset w - I_C(w), \quad \forall w \in Y.$$

(iii) *The binary relation $\leq^C$ is antisymmetric if and only if*

$$\bigcup_{y \in Y} \bigcup_{d \in D_C(y)} [\{d\} \cap (-D_C(y + d))] \subset \{0\},$$

or

$$\bigcup_{y \in Y} \bigcup_{d \in I_C(y)} [\{d\} \cap (-I_C(y - d))] \subset \{0\}.$$

(iv) *The binary relation $\leq^C$ is compatible with addition if and only if D_C (or I_C) is a constant mapping.*

(v) *The binary relation* $\leq^C$ *is compatible with scalar multiplication if and only if*

$$\alpha D_C(y) \subset D_C(\alpha y), \quad \forall y \in Y, \forall \alpha > 0$$

or

$$\alpha I_C(y) \subset I_C(\alpha y), \quad \forall y \in Y, \forall \alpha > 0.$$

Theorem 3.5 implies the following properties related to $\leq_1^{\mathcal{D}}$ and $\leq_2^{\mathcal{D}}$.

Corollary 3.8. *Consider a set-valued mapping* $\mathcal{D} : Y \rightrightarrows Y$. *We have:*

(i) *For each* $i \in \{1, 2\}$, $\leq_i^{\mathcal{D}}$ *is reflexive if and only if* $0 \in \mathcal{D}(y)$, *for all* $y \in Y$.

(ii) *The relation* $\leq_1^{\mathcal{D}}$ *(correspondingly,* $\leq_2^{\mathcal{D}}$*) is transitive if and only if*

$$d + \mathcal{D}(y + d) \subset \mathcal{D}(y)$$
$$(\textit{correspondingly, } d + \mathcal{D}(y - d) \subset \mathcal{D}(y)), \quad \forall y \in Y, \forall d \in \mathcal{D}(y).$$

(iii) *The relation* $\leq_1^{\mathcal{D}}$ *(correspondingly* $\leq_2^{\mathcal{D}}$*) is antisymmetric if and only if*

$$d \notin -\mathcal{D}(y + d)$$
$$(\textit{correspondingly, } d \notin -\mathcal{D}(y - d)), \quad \forall y \in Y, \forall d \in \mathcal{D}(y) \backslash \{0\}.$$

(iv) *For each* $i \in \{1, 2\}$, $\leq_i^{\mathcal{D}}$ *is compatible with addition if and only if* $\mathcal{D}$ *is a constant mapping.*

(v) *For each* $i \in \{1, 2\}$, $\leq_i^{\mathcal{D}}$ *is compatible with scalar multiplication if and only if*

$$\alpha \mathcal{D}(y) \subset \mathcal{D}(\alpha y), \quad \forall y \in Y, \forall \alpha > 0.$$

3.1.2 Efficient point and optimal solution concepts

This section is divided into two parts: The concepts of minimal points and non-dominated points (collectively referred to as efficient points) of a set are presented in the first part, and the concepts of optimal solutions for vector optimization problems with two-variable domination structures are presented in the second part. Related properties are also examined in detail.

a) Efficient points of a set

Let Ω be a nonempty subset of Y and $C : Y \times Y \rightrightarrows Y$ be a two-variable ordering mapping. We define two concepts related to the efficient points of the set Ω according to the relation $\leq^C$ as follows.

Definition 3.10. A point $\bar{y} \in \Omega$ is called a

(a) $\leq^C$-minimal point of Ω, denoted by $\bar{y} \in \mathrm{Min}(\Omega, C)$, if

$$y \in \Omega,\ y \leq^C \bar{y} \quad \Rightarrow \quad \bar{y} \leq^C y;$$

(b) $\leq^C$-nondominated point of Ω, denoted by $\bar{y} \in \mathrm{ND}(\Omega, C)$, if there does not exist $y \in \Omega \setminus \{\bar{y}\}$ satisfying $y \leq^C \bar{y}$;

(c) strongly $\leq^C$-nondominated point of Ω, denoted by $\bar{y} \in \mathrm{SND}(\Omega, C)$, if $\bar{y} \leq^C y$, for all $y \in \Omega \setminus \{\bar{y}\}$;

(d) Assume that $\mathrm{dom}(\mathrm{cor}\, C) = Y \times Y$. Then, $\bar{y}$ is called a weakly $\leq^C$-nondominated point of Ω, denoted by $\bar{y} \in \mathrm{WND}(\Omega, C)$, if $\bar{y} \in \mathrm{ND}(\Omega, \mathrm{cor}\, C)$.

In the following parts, we denote

$$I_C^\Omega(y) := \{d \in I_C(y) : y - d \in \Omega\}, \quad \forall y \in \Omega.$$

Remark 3.12. Clearly, $\mathrm{ND}(\Omega, C_2) \subset \mathrm{ND}(\Omega, C_1)$ if the two-variable ordering mappings C_1 and C_2 satisfy $\mathrm{gph}\, C_1 \subset \mathrm{gph}\, C_2$.

The following theorem shows the relationships between different concepts of efficient points of a set.

Theorem 3.13. *Let $\bar{y} \in \Omega$.*

(i) *If $\bar{y}$ is a $\leq^C$-nondominated point of Ω, then $\bar{y}$ is a weakly $\leq^C$-nondominated point of Ω.*

(ii) *Assume that $C(y, \bar{y}) \cap (-C(\bar{y}, y)) \subset \{0\}$ for all $y \in \Omega \setminus \{\bar{y}\}$. Then, if $\bar{y}$ is a strongly $\leq^C$-nondominated point of Ω, $\bar{y}$ is also a $\leq^C$-nondominated point of Ω. Furthermore, there exists at most one strongly $\leq^C$-nondominated point of Ω.*

(iii) *If $\bar{y}$ is a $\leq^C$-nondominated point of Ω, then it is also a $\leq^C$-minimal point of Ω. Conversely, if $\bar{y}$ is a $\leq^C$-minimal point of Ω and*

$$\bigcup_{d \in I_C^\Omega(\bar{y}) \setminus \{0\}} [\{d\} \cap (-I_C^\Omega(\bar{y} - d))] = \emptyset,$$

then it is also a $\leq^C$-nondominated point of Ω.

The following result allows us to reduce the search for $\leq^C$-nondominated points of a set to a search on the boundary of that set.

Theorem 3.15. *Let $\bar{y} \in Y$.*

(i) *If* $\bar{y}$ *is a* $\leq^C$*-nondominated point of* Ω *and there exists a net* $(y_i) \subset Y\backslash\{\bar{y}\}$ *satisfying* $y_i \to \bar{y}$ *and*

$$\bar{y} - y_i \in C(y_i, \bar{y}), \quad \forall i$$

then $\bar{y} \in \partial M$.

(ii) *If* $\bar{y}$ *is a weakly* $\leq^C$*-nondominated point of* Ω *and there exists a net* $(y_i) \subset Y\backslash\{\bar{y}\}$ *satisfying* $y_i \to \bar{y}$ *and*

$$\bar{y} - y_i \in \operatorname{cor} C(y_i, \bar{y}), \quad \forall i$$

then $\bar{y} \in \partial\Omega$.

b) Optimal solution concepts for vector optimization problems

Consider the constrained vector optimization problem

$$\min_C\{f(x) : x \in S\}, \tag{P}$$

where $f : X \to Y$, S is a nonempty subset of X, and $C : Y \times Y \rightrightarrows Y$ defines a two-variable domination structure $\leq^C$ on Y as in Definition 3.1. We denote $\mathcal{N}(\bar{x})$ as the set of all neighborhoods of $\bar{x} \in X$. Next, we define the optimal solution concepts for problem (P).

Definition 3.17. A point $\bar{x} \in S$ is called a

(a) local C-minimal solution of problem (P), denoted by $\bar{x} \in \mathrm{LMin}(f, S, C)$, if there exists $U \in \mathcal{N}(\bar{x})$ such that

$$x \in U \cap S, \quad f(x) \leq^C f(\bar{x}) \Rightarrow f(\bar{x}) \leq^C f(x). \tag{3.23}$$

(b) local C-nondominated solution of (P), denoted by $\bar{x} \in \mathrm{LND}(f, S, C)$, if there exists $U \in \mathcal{N}(\bar{x})$ such that

$$x \in U \cap S, \quad f(x) \leq^C f(\bar{x}) \Rightarrow f(x) = f(\bar{x}). \tag{3.24}$$

(c) local strongly C-nondominated solution of problem (P), denoted by $\bar{x} \in \mathrm{LSND}(f, S, C)$, if there exists $U \in \mathcal{N}(\bar{x})$ such that

$$f(\bar{x}) \leq^C f(x), \quad \forall x \in U \cap (S\backslash\{\bar{x}\}). \tag{3.25}$$

(d) local weakly C-nondominated solution of problem (P), denoted by $\bar{x} \in \mathrm{LWND}(f, S, C)$, if $\bar{x} \in \mathrm{LND}(f, S, \mathrm{cor}C)$.

If condition (3.23) (correspondingly, (3.24), (3.25)) holds for all $x \in S \setminus \{\bar{x}\}$, then $\bar{x}$ is called a global C-minimal solution (correspondingly, global C-nondominated solution, global strongly C-nondominated solution) of problem (P), denoted by $\bar{x} \in \mathrm{GMin}(f, S, C)$ (correspondingly, $\bar{x} \in \mathrm{GND}(f, S, C)$, $\bar{x} \in \mathrm{GSND}(f, S, C)$). Similarly, $\bar{x}$ is called a global weakly C-nondominated solution of problem (P), denoted by $\bar{x} \in \mathrm{GWND}(f, S, C)$, if $\bar{x} \in \mathrm{GND}(f, S, \mathrm{cor}C)$.

The concepts of local C-minimal solutions and local C-nondominated solutions encompass the known optimal solution concepts related to variable domination structures defined by a single-variable set-valued mapping $\mathcal{D} : Y \rightrightarrows Y$ via two two-variable domination structures constructed based on $\mathcal{D}$, namely $C_1^{\mathcal{D}}$ and $C_2^{\mathcal{D}}$. Specifically, by replacing $\leq^C$ with $C_1^{\mathcal{D}}$ and $C_2^{\mathcal{D}}$, we obtain the following concepts.

Definition 3.19. Consider a single-variable domination structure $\mathcal{D} : Y \rightrightarrows Y$. For $i = 1, 2$, a point $\bar{x} \in S$ is called

(a) a local $\leq_i^{\mathcal{D}}$-minimal solution of problem (P), denoted by $\bar{x} \in \mathrm{LMin}(f, S, \leq_i^{\mathcal{D}})$, if there exists $U \in \mathcal{N}(\bar{x})$ such that the following inclusion is satisfied:

$$x \in U \cap S, \quad f(x) \leq_i^{\mathcal{D}} f(\bar{x}) \Rightarrow f(\bar{x}) \leq_i^{\mathcal{D}} f(x).$$

(b) a local $\leq_i^{\mathcal{D}}$-nondominated solution of problem (P) if there exists $U \in \mathcal{N}(\bar{x})$ such that

$$x \in U \cap S, \quad f(x) \leq_i^{\mathcal{D}} f(\bar{x}) \Rightarrow f(x) = f(\bar{x}).$$

In this case, we denote $\bar{x} \in \mathrm{LND}(f, S, \leq_i^{\mathcal{D}})$.

The global solution concepts corresponding to the local solution concepts mentioned above are denoted by $\mathrm{GMin}(f, S, \leq_i^{\mathcal{D}})$ and $\mathrm{GND}(f, S, \leq_i^{\mathcal{D}})$, respectively.

The following theorem establishes the connections between different solution concepts of problem (P).

Theorem 3.21. (i) *Every local C-nondominated solution of problem* (P) *is also a local weakly C-nondominated solution of problem* (P).

(ii) *Assume that there exists $U \in \mathcal{N}(\bar{x})$ such that*

$$C(y, f(\bar{x})) \cap (-C(f(\bar{x}), y)) \subset \{0\}, \quad \forall y \in f(U \cap S) \backslash \{f(\bar{x})\}.$$

Then, if $\bar{x}$ is a local strongly C-nondominated solution of problem (P), *it is also a local C-nondominated solution of problem* (P).

(iii) *Every local C-nondominated solution of problem* (P) *is also a local C-minimal solution. Conversely, if $\bar{x} \in S$ is a local $\leq^C$-minimal solution and there exists $U \in \mathcal{N}(\bar{x})$ such that*

$$\bigcup_{d \in I_C^{f(S\cap U)}(f(\bar{x}))\setminus\{0\}} [\{d\} \cap (-I_C^{f(S\cap U)}(f(\bar{x}) - d))] = \emptyset,$$

then it is also a local C-nondominated solution.

Next, we investigate the relationship between the optimal C-solution concepts of problem (P) and the corresponding concepts related to single-variable domination structures.

Let $\bar{x} \in S$. For each $V \in \mathcal{N}(\bar{x})$, define the set-valued mappings $\mathcal{D}^i_{1,V}, \mathcal{D}^u_{1,V}, \mathcal{D}^u_{2,V} : Y \rightrightarrows Y$ and $\widehat{C}_V : Y \times Y \rightrightarrows Y$ as follows:

$$\begin{aligned}
\mathcal{D}^i_{1,V}(y) &= \bigcap_{x\in V\cap S} C(f(x), y),\\
\mathcal{D}^u_{1,V}(y) &= \bigcup_{x\in V\cap S} C(f(x), y),\\
\mathcal{D}^u_{2,V}(y) &= \bigcup_{x\in V\cap S} C(y, f(x)),\\
\widehat{C}_V(y_1, y_2) &= \big(C(y_1, y_2) \setminus (-\mathcal{D}^u_{2,V}(y_2))\big) \cup \{0\}.
\end{aligned}$$

Theorem 3.23. (i) *If $\bar{x} \in \bigcup_{V\in\mathcal{N}(\bar{x})} \mathrm{LND}(f, S, \leq_2^{\mathcal{D}^u_{1,V}})$, then $\bar{x} \in \mathrm{LND}(f, S, C)$.*

(ii) *Assume that $\operatorname{dom} \mathcal{D}^i_{1,V} = Y$ for all $V \in \mathcal{N}(\bar{x})$. Then, if $\bar{x} \in \mathrm{LND}(f, S, C)$, we have $\bar{x} \in \bigcup_{V\in\mathcal{N}(\bar{x})} \mathrm{LND}(f, S, \leq_2^{\mathcal{D}^i_{1,V}})$.*

(iii) If $\bar{x} \in \mathrm{LMin}(f, S, C)$, then we have $\bar{x} \in \bigcup_{V\in\mathcal{N}(\bar{x})} \mathrm{LND}(f, S, \widehat{C}_V)$.

3.2 Solution characterizations using nonlinear scalarization functions

In this section, necessary and sufficient conditions for local nondominated solutions will be obtained via the scalarization method. This is an important mathematical tool when studying vector optimization problems. The scalarization tech-

nique usually relies on scalarization functions, among which the nonlinear scalarization function of the following form is particularly crucial:

$$\varphi_{K,q}(y) = \inf\{t \in \mathbb{R} : y \in tq - K\}.$$

This function is widely applied to obtain results for nonconvex vector optimization problems, where the image space Y with the partial order relation $\leq_K$ is defined by a fixed convex pointed cone K. Here, we will use the scalarization method to characterize $\leq^C$-nondominated/weakly nondominated points of a set Ω and local C-nondominated/weakly nondominated solutions of problem (P) through an extended form of the function $\varphi_{K,q}$. This scalarization function allows the direction q to vary by using a mapping $q : Y \times Y \to Y$ instead of a fixed vector q. The obtained results offer two novel aspects. First, generalized scalarization results applicable to the two-variable ordering mapping C are established. Second, in the special case of considering a single-variable domination structure, the obtained results relax the assumptions of the problem. The generalized Gerstewitz scalarization function is defined as follows.

Definition 3.25. Consider a set-valued mapping $C : Y \times Y \rightrightarrows Y$ satisfying $\operatorname{dom} C = Y \times Y$, a function $q : Y \times Y \to Y\backslash\{0\}$, and a point $\bar{y} \in Y$. The nonlinear scalarization function $\varphi^C_{\bar{y},q} : Y \to \overline{\mathbb{R}}$ is defined as follows:

$$\varphi^C_{\bar{y},q}(y) := \inf \Lambda^C_q(y, \bar{y}),$$

where $\Lambda^C_q : Y \times Y \rightrightarrows \mathbb{R}$,

$$\Lambda^C_q(y_1, y_2) := \{t \in \mathbb{R} : y_2 + tq(y_1, y_2) - y_1 \in C(y_1, y_2)\}$$

with the convention $\inf \emptyset = +\infty$.

Given a nonempty set $E \subset Y$ and a nonzero vector $q \in Y$, the closure of E along the direction q (hereafter called the q-vector closure of E) is the set:

$$\operatorname{vcl}_q E := \{y \in Y : \forall \lambda > 0\, \exists\, \lambda' \in [0, \lambda] \text{ such that } y + \lambda' q \in E\}.$$

We say that the set E is closed along the direction q if $\operatorname{vcl}_q E = E$. The boundary along the direction q of E is the set $\operatorname{bd}_q E := \operatorname{vcl}_q E \backslash \operatorname{int}_{-q} E$. Recall that the recession cone of a set E is the convex cone:

$$0^+E := \{y \in Y : E + \mathbb{R}_+ y = E\}.$$

The following lemma summarizes some properties of the vector closure of a set.

Lemma 3.26. *We have:*

(i) $y \in \mathrm{vcl}_q E$ *if and only if there exists a sequence* $(t_n) \subset \mathbb{R}_+$ *such that* $t_n \to 0$ *and* $y + t_n q \in E$ *for all* n.

(ii) $E \subset \mathrm{vcl}_q E \subset \mathrm{vcl}_q E + \mathbb{R}_+ q = \mathrm{vcl}_q(E + \mathbb{R}_+ q)$.

(iii) $\mathrm{vcl}_q(\mathrm{vcl}_q E) = \mathrm{vcl}_q E$ *and* $\mathrm{vcl}_q(\mathrm{vcl}_q E + \mathbb{R}_+ q) = \mathrm{vcl}_q(E + \mathbb{R}_+ q)$.

(iv) $Y \backslash \mathrm{vcl}_q E = \mathrm{int}_q(Y \backslash E)$.

(v) *If* $q \in 0^+ E$, *then*

$$E + (0, +\infty)q = \mathrm{int}_{-q} E,$$
$$\mathrm{vcl}_q E + [0, +\infty)q = \mathrm{vcl}_q E.$$

(vi) *If* $q \in -\mathrm{cor}(0^+ E) \backslash \{0\}$, *then* $\mathrm{cor} E = \mathrm{int}_q E$.

Now we consider some properties of the scalarization function $\varphi^C_{\bar{y},q}$. Given $C : Y \times Y \rightrightarrows Y$ and $q : Y \times Y \to Y \backslash \{0\}$, $\mathrm{vcl}_q C$, $\mathrm{bd}_q C$, and $\mathrm{vcl}_q C + \mathbb{R}_+ q$ denote the set-valued mappings from $Y \times Y$ into Y given by $(\mathrm{vcl}_q C)(y_1, y_2) = \mathrm{vcl}_{q(y_1,y_2)} C(y_1, y_2)$, $(\mathrm{bd}_q C)(y_1, y_2) = \mathrm{bd}_{q(y_1,y_2)} C(y_1, y_2)$, and

$$(\mathrm{vcl}_q C + \mathbb{R}_+ q)(y_1, y_2) = (\mathrm{vcl}_q C)(y_1, y_2) + \mathbb{R}_+ q(y_1, y_2).$$

Lemma 3.27. *The following assertions hold:*

(i) $\varphi^C_{\bar{y},q} = \varphi^{C'}_{\bar{y},q}$ *for any set-valued mapping* $C' : Y \times Y \rightrightarrows Y$ *satisfying*

$$\mathrm{gph}\, C \subset \mathrm{gph}\, \mathrm{vcl}_q C' \subset \mathrm{gph}(\mathrm{vcl}_q C + \mathbb{R}_+ q).$$

In particular, $\varphi^C_{\bar{y},q} = \varphi^{\mathrm{vcl}_q C}_{\bar{y},q} = \varphi^{\widetilde{C}}_{\bar{y},q}$, *where* $\widetilde{C} := \mathrm{vcl}_q C + \mathbb{R}_+ q$.

(ii) *For all* $y \in Y$ *and* $s \in \mathbb{R}$, *we have*

$$\varphi^C_{\bar{y},q}(y) < s \Leftrightarrow y - \bar{y} \in (-\infty, s)q(y, \bar{y}) - C(y, \bar{y}).$$

(iii) *For all* $y \in Y$ *and* $s \in \mathbb{R}$, *we have*

$$\varphi^C_{\bar{y},q}(y) \leq s \Leftrightarrow y - \bar{y} \in (-\infty, s]q(y, \bar{y}) - \mathrm{vcl}_{q(y,\bar{y})} C(y, \bar{y}).$$

Next, we establish necessary and sufficient conditions for nondominated and weakly nondominated points of a set under the domination structure $\leq^C$.

Theorem 3.31. *Let* Ω *be a nonempty subset of* Y *and* $\bar{y} \in \Omega$.

(i) *Assume that* $q(y, \bar{y}) \in 0^+ C(y, \bar{y})$ *for all* $y \in Y$ *and* $0 \in \mathrm{bd}_{q(\bar{y},\bar{y})} C(\bar{y}, \bar{y})$. *Then,* $\bar{y} \in \mathrm{ND}(\Omega, \mathrm{int}_{-q} C)$ *if and only if* $\bar{y} \in \mathrm{argmin}_\Omega\, \varphi^C_{\bar{y},q}$.

(ii) *Assume that* $q(y,\bar{y}) \in \operatorname{cor} 0^+C(y,\bar{y})$ *for all* $y \in Y\backslash\{\bar{y}\}$, $q(\bar{y},\bar{y}) \in 0^+C(\bar{y},\bar{y})$, *and* $0 \in \operatorname{bd}_{q(\bar{y},\bar{y})}C(\bar{y},\bar{y})$. *Then,* $\bar{y} \in \operatorname{WND}(\Omega, C)$ *if and only if* $\bar{y} \in \operatorname{argmin}_\Omega \varphi^C_{\bar{y},q}$.

(iii) *Assume that* $q(y,\bar{y}) \in 0^+C(y,\bar{y})$ *for all* $y \in Y$ *and* $0 \in \operatorname{bd}_{q(\bar{y},\bar{y})}C(\bar{y},\bar{y})$. *Then* $\bar{y} \in \operatorname{ND}(\Omega, \operatorname{vcl}_q C)$ *if and only if* $\bar{y} \in \operatorname{argsmin}_\Omega \varphi^C_{\bar{y},q}$.

(iv) *Assume that* $C(y,\bar{y})$ *is closed along the direction* $q(y,\bar{y})$ *for all* $y \in Y\backslash\{\bar{y}\}$, $q(y,\bar{y}) \in 0^+C(y,\bar{y})$ *for all* $y \in Y$, *and* $0 \in \operatorname{bd}_{q(\bar{y},\bar{y})}C(\bar{y},\bar{y})$. *Then,* $\bar{y} \in \operatorname{ND}(\Omega, C)$ *if and only if* $\bar{y} \in \operatorname{argsmin}_\Omega \varphi^C_{\bar{y},q}$.

(v) *Assume that* $q(y,\bar{y}) \in 0^+C(y,\bar{y})$ *for all* $y \in Y$ *and* $0 \in \operatorname{bd}_{q(\bar{y},\bar{y})}C(\bar{y},\bar{y})$. *Then, if* $\bar{y} \in \operatorname{argmin}_\Omega \varphi^C_{\bar{y},q}\backslash\operatorname{argsmin}_\Omega \varphi^C_{\bar{y},q}$, *we have*

$$\emptyset \neq \{y \in M\backslash\{\bar{y}\} : \bar{y} \in y + \operatorname{bd}_{q(y,\bar{y})}C(y,\bar{y})\} = \{y \in \Omega\backslash\{\bar{y}\} : \varphi^C_{\bar{y},q}(y) = 0\}.$$

The next part presents results regarding the existence of nondominated points of the set Ω according to the two-variable domination structure generated by the set-valued mapping $\operatorname{int}_{-q}C$. For each $C : Y \times Y \rightrightarrows Y$, $q : Y \times Y \to Y$, $\bar{y} \in Y$, and $r \in \mathbb{R}$, we denote the set-valued mapping $(\operatorname{vcl}_q C + rq)_{\bar{y}} : Y \rightrightarrows Y$ defined as follows:

$$(\operatorname{vcl}_q C + rq)_{\bar{y}}(y) := \operatorname{vcl}_{q(y,\bar{y})}C(y,\bar{y}) + rq(y,\bar{y}) \quad \forall y \in Y.$$

Lemma 3.35. *Consider a point* $\bar{y} \in Y$. *Assume that* $q(y,\bar{y}) \in 0^+C(y,\bar{y})$ *and the graph of the mapping* $(\operatorname{vcl}_q C + rq)_{\bar{y}}$ *is a closed set for all* $y \in Y$ *and* $r \in \mathbb{R}$. *Then* $\varphi^C_{\bar{y},q}$ *is lower semicontinuous.*

Theorem 3.36. *Let* $\Omega \subset Y$ *be a nonempty compact set. Assume that* $q(y_1,y_2) \in 0^+C(y_1,y_2)$, $0 \in \operatorname{bd}_{q(y,y)}C(y,y)$, *and the graph of* $(\operatorname{vcl}_q C + rq)_y$ *is closed for all* $y_1, y_2, y \in Y$ *and* $r \in \mathbb{R}$. *Furthermore, assume that the function* $\varphi^C_{y,q}$ *is proper for all* $y \in Y$. *Then,* $\operatorname{argmin}_\Omega \varphi^C_{y,q} \neq \emptyset$ *for all* $y \in Y$ *and*

$$\operatorname{ND}(\Omega, \operatorname{int}_{-q}C) = \bigcup_{y\in\Omega}[\operatorname{argmin}_\Omega \varphi^C_{y,q}] \cap \{y\}.$$

We conclude this section with Corollary 3.37. This result is obtained by applying Theorem 3.31 to characterize local solutions of problem (P). We denote the set of local solutions (correspondingly, strict local solutions) of the scalar optimization problem with objective function $g : X \to \overline{\mathbb{R}}$ and constraint set $S \subset X$ by $\operatorname{arglmin}_S g$ (correspondingly $\operatorname{argslmin}_S g$), that is:

$$\bar{x} \in \operatorname{arglmin}_S g :\Leftrightarrow \bar{x} \in \bigcup_{V\in\mathcal{N}(\bar{x})} \operatorname{argmin}_{S\cap V} g,$$

$$(\text{correspondingly,} \quad \bar{x} \in \operatorname{argslmin}_S g :\Leftrightarrow \bar{x} \in \bigcup_{V \in \mathcal{N}(\bar{x})} \operatorname{argsmin}_{S \cap V} g).$$

In addition, related to problem (P), the level set of f at $y \in Y$ is denoted by $\mathrm{L}(f, y)$, that is:

$$\mathrm{L}(f, y) := \{x \in X : f(x) = y\}.$$

Corollary 3.37. *Consider problem* (P) *and let* $\bar{x} \in S$.

(i) *Assume that* $q(y, f(\bar{x})) \in 0^+C(y, f(\bar{x}))$ *for all* $y \in Y$ *and* $0 \in \operatorname{bd}_{q(f(\bar{x}), f(\bar{x}))} C(f(\bar{x}), f(\bar{x}))$. *Then,*

$$\bar{x} \in \mathrm{LND}(f, S, \operatorname{int}_{-q} C) \Leftrightarrow \bar{x} \in \operatorname{arglmin}_S (\varphi^C_{f(\bar{x}), q} \circ f).$$

(ii) *Assume that* $q(f(\bar{x}), f(\bar{x})) \in 0^+C(f(\bar{x}), f(\bar{x}))$, $q(y, f(\bar{x})) \in \operatorname{cor} 0^+C(y, f(\bar{x}))$ *for all* $y \in Y \backslash \{f(\bar{x})\}$, *and* $0 \in \operatorname{bd}_{q(f(\bar{x}), f(\bar{x}))} C(f(\bar{x}), f(\bar{x}))$. *Then,*

$$\bar{x} \in \mathrm{LWND}(f, S, C) \Leftrightarrow \bar{x} \in \operatorname{arglmin}_S (\varphi^C_{f(\bar{x}), q} \circ f).$$

(iii) *Assume that* $q(y, f(\bar{x})) \in 0^+C(y, f(\bar{x}))$ *for all* $y \in Y$ *and* $0 \in \operatorname{bd}_{q(f(\bar{x}), f(\bar{x}))} C(f(\bar{x}), f(\bar{x}))$. *Then,*

$$\bar{x} \in \mathrm{LND}(f, S, \operatorname{vcl}_q C) \Leftrightarrow \bar{x} \in \operatorname{argslmin}_{(S \backslash \mathrm{L}(f, f(\bar{x}))) \cup \{\bar{x}\}} (\varphi^C_{f(\bar{x}), q} \circ f).$$

(iv) *Assume that* $q(y, f(\bar{x})) \in 0^+C(y, f(\bar{x}))$ *for all* $y \in Y$, $C(y, f(\bar{x}))$ *is closed along the direction* $q(y, f(\bar{x}))$ *for all* $y \in Y \backslash \{f(\bar{x})\}$, *and* $0 \in \operatorname{bd}_{q(f(\bar{x}), f(\bar{x}))} C(f(\bar{x}), f(\bar{x}))$. *Then,*

$$\bar{x} \in \mathrm{LND}(f, S, C) \Leftrightarrow \bar{x} \in \operatorname{argslmin}_{(S \backslash \mathrm{L}(f, f(\bar{x}))) \cup \{\bar{x}\}} (\varphi^C_{f(\bar{x}), q} \circ f).$$

Conclusion

The dissertation has achieved several new and meaningful results in the fields of minimax optimization and vector optimization.

First, the dissertation has established necessary and sufficient optimality conditions for discrete minimax optimization problems in both convex and nonconvex cases. The optimality conditions are formulated through the concepts of subdifferentials and corresponding normal cones from convex analysis and variational analysis, under the metric qualification condition and continuity (or Lipschitz continuity) condition. In particular, the subdifferential calculus rule for the maximum function under the metric qualification condition is constructed and plays a key role in deriving the optimality results, which represents a new contribution that has not been previously established.

In addition, the dissertation has carried out sensitivity analysis for classes of parametric minimax optimization problems in both convex and nonconvex cases. Formulae for evaluating the subdifferential of the optimal value function are established, thereby clarifying the dependence of the optimal value on the problem parameters under suitable qualification assumptions.

Finally, the dissertation has systematically studied two-variable domination structures in linear spaces, clarifying the fundamental properties of this structure as well as its connections with conventional variable domination structures. On this basis, optimal solution concepts for vector optimization problems under two-variable domination structures are constructed and compared with existing concepts and results. The dissertation has also established necessary and sufficient optimality conditions for vector optimization problems with two-variable domination structures via nonlinear scalarization methods, which are then applied to the special case of conventional variable domination structures. The new results obtained not only generalize but also improve upon existing results.

The findings of this dissertation not only contribute to completing the theoretical framework for the considered problems but also open up many future research directions in minimax optimization and vector optimization with two-variable domination structures.