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SOME ITERATIVE METHODS FOR SOLVING THE SPLIT COMMON SOLUTION PROBLEM FOR MONOTONE OPERATOR EQUATIONS WITH MULTIPLE OUTPUT SETS

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List of published works related to the thesis

(CT1) Reich, S., Tuyen, T. M., & Huyen, P. T. V. (2024). Inertial proximal point algorithms for solving a class of split feasibility problems. *Journal of Optimization Theory and Applications*, **200**(3), 951–977.

(CT2) Ha, N. S., Tuyen, T. M., & Huyen, P. T. V. (2024). A new approach to solving the split common solution problem for monotone operator equations in Hilbert spaces. *Bulletin of the Brazilian Mathematical Society, New Series*, **55**, Article 29.

(CT3) Huyen, P. T. V., Ha, N. S., & Tuyen, T. M. (2025). Two generalized cyclic projection algorithms for solving a class of the split feasibility problem in real Hilbert spaces. *Carpathian Journal of Mathematics*, **41**(2), 525–541.

Introduction

The Operator Equation Problem, abbreviated as (OEQ), is one of the fundamental and important models in modern mathematics. Many problems in analysis and applications can be reduced to solving a general operator equation of the form

$$A(x) = f, \quad x \in U,\ f \in V, \tag{OEQ}$$

where A is an operator from the space U into the space V.

In many areas of modern applied mathematics, one often encounters problems in which the solution must simultaneously satisfy several constraints posed in different spaces. One of the typical mathematical models for this type of problem is the *Split Feasibility Problem* in finite-dimensional Hilbert spaces (SFP, for short), which was first introduced by *Yair Censor* and *Tommy Elfving* in 1994 for inverse problem models. Problem (SFP) can be formulated as follows: Let C and Q be nonempty, closed, and convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $T:\ \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator.

$$\text{Find an element } p \in C \text{ such that } T(p) \in Q. \tag{SFP}$$

Since its introduction, problem (SFP) has become an important research topic and has been widely applied in many practical fields such as signal processing and image reconstruction, intensity-modulated radiation therapy for optimizing treatment planning, equilibrium problems in economics and game theory, as well as several other applications.

The split feasibility problem and its related models have been increasingly extended and generalized. In 2020, Reich et al. studied and introduced a general model of the split feasibility problem, called the *Split Feasibility Problem with Multiple Output Sets* (SFP-MOS, for short). The problem is formulated as follows.

Let $\mathcal{H}$ and $\mathcal{H}_i$, $i = 1, 2, \ldots, N$, be real Hilbert spaces, and let $T_i:\ \mathcal{H} \to \mathcal{H}_i$, $i = 1, 2, \ldots, N$, be bounded linear operators. Let C and Q_i be nonempty, closed, and convex subsets of $\mathcal{H}$ and $\mathcal{H}_i$, $i = 1, 2, \ldots, N$, respectively. The problem is to

$$\text{find an element } p \in \mathcal{H} \text{ such that } p \in C \text{ and } T_i(p) \in Q_i, \tag{SFP-MOS}$$

for all $i = 1, 2, \ldots, N$.

In the case $N = 1$, problem (SFP-MOS) reduces to problem (SFP). Several related classes of problems, such as the split common fixed point problem with multiple output sets and the split common null point problem with multiple output sets, have also been investigated.

Although it has been proposed only recently, the class of split feasibility problems with *multiple output sets* has attracted considerable attention from the research community. Evidence of this growing interest is the increasing number of scientific works published along this research direction. These studies mainly focus on two aspects: first, extending and generalizing the problem model to broader classes of mappings or more general spaces; second, proposing, analyzing, and improving iterative algorithms in order to enhance the convergence rate and stability of the methods.

Motivated by the problem models (OEQ), (SFP), and (SFP-MOS), in 2023, in paper (CT1), we introduced and investigated a new problem model called the "***Split Common Solution Problem for Monotone Operator Equations with Multiple Output Sets***" (SCSP-MOE, for short) in Hilbert spaces, together with several related problems.

The problem (SCSP-MOE) in Hilbert spaces is formulated as follows. Let $\mathcal{H}$ and $\mathcal{H}_i$, $i = 1, 2, \ldots, N$, be real Hilbert spaces. Let $A: \mathcal{H} \to \mathcal{H}$ and $A_i: \mathcal{H}_i \to \mathcal{H}_i$, $i = 1, 2, \ldots, N$, be monotone operators. Let $T_i: \mathcal{H} \to \mathcal{H}_i$, $i = 1, 2, \ldots, N$, be bounded linear operators. Let f and f_i be given elements in $\mathcal{H}$ and $\mathcal{H}_i$, respectively, $i = 1, 2, \ldots, N$. The problem is to

$$\text{find an element } p \in \Omega = \Omega^A \cap \Big(\bigcap_{i=1}^{N} \Omega^{A_i} \Big) \neq \emptyset, \qquad \text{(SCSP-MOE)}$$

where

$$\Omega^A = \{x \in \mathcal{H} \mid A(x) = f\},$$
$$\Omega^{A_i} = \{x \in \mathcal{H} \mid A_i(T_i(x)) = f_i\}, \quad \text{for all } i = 1, 2, \ldots, N.$$

If $S: \mathcal{H} \to \mathcal{H}$ is a nonexpansive mapping, then $A = I - S$ is $\frac{1}{2}$-inverse strongly monotone. Consequently, in some special cases, the (SCSP-MOE) reduces to the *split common fixed point problem with multiple output sets*, the *split feasibility problem with multiple output sets*, or the *split common null point problem with multiple output sets*.

Therefore, it can be seen that problem (SCSP-MOE) encompasses several classes of split-type problems. Hence, the study of this problem model, together with

methods for finding its solutions, constitutes an important research topic with significant scientific relevance. For this reason, we choose this problem as the main subject of investigation in this dissertation.

Besides the research direction devoted to extending and generalizing models of the split feasibility problem, the development of methods and algorithms for solving this class of problems has attracted considerable attention from many authors.

According to *Hadamard* (1932), most split-type problems, especially those involving inverse operators, are typically *ill-posed problems*. To overcome this difficulty, *Tikhonov* (1943) proposed the *Tikhonov regularization method* – a classical technique that still retains its fundamental importance today. The main idea of this method is that, instead of solving the ill-posed problem

$$Ax = b,$$

one considers the well-posed minimization problem

$$\min_{x \in H} \ \|Ax - b\|^2 + \lambda \|x\|^2,$$

where A is a linear operator and $\lambda > 0$ is called the *regularization parameter*. As $\lambda \to 0$, the regularized solution x_λ approaches the true solution of the original problem. The Tikhonov regularization method not only provides a fundamental framework for many modern techniques in solving inverse problems but has also been combined with several modern iterative methods in optimization and split feasibility problems in order to improve stability and achieve better convergence behavior.

One of the most popular methods for solving the split feasibility problem is the CQ method, which was first introduced by *C. Byrne*. In general, the CQ algorithm and several of its extensions only guarantee weak convergence in infinite-dimensional real Hilbert spaces, with the stepsize chosen depending on the norm of the associated linear operator. In practice, however, computing the norm of an operator is often not straightforward. To overcome the limitation related to the computation of the operator norm in the CQ algorithm, *self-adaptive algorithms* have been proposed. The main advantage of these algorithms is that the stepsizes are constructed adaptively and do not depend on the norm of the associated operator.

The Proximal Point Algorithm (PPA) is also one of the important methods for solving the class of split feasibility problems. This algorithm was first studied by *Martinet* for minimizing convex functionals in a Hilbert space $\mathcal{H}$. In 1976, *Rockafellar* applied Martinet's proximal point method to find the *zeros of a maximal*

monotone operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ in a Hilbert space $\mathcal{H}$. However, the proximal point method generally guarantees only weak convergence in infinite-dimensional Hilbert spaces. To obtain strong convergence, the proximal point method has been combined with several other techniques, such as the Tikhonov regularization method, the Halpern iterative method, the viscosity approximation method, the shrinking projection method, the hybrid projection method, and some other approaches.

Another extension of the proximal point method is to incorporate an inertial term into the iterative scheme of the algorithm in order to accelerate the convergence rate. *Inertial algorithms* originate from classical mechanics. The first work that laid the foundation for inertial algorithms is the paper by *Polyak* (1964). The inertial idea was later standardized and further developed by *Polyak* (1987) in the well-known *Heavy Ball* method, whose typical iterative scheme is given by

$$x^{n+1} = x^n - \alpha \nabla f(x^n) + \beta(x^n - x^{n-1}),$$

where $\alpha > 0$ and $\beta \in [0, 1)$. Thanks to the "inertial" component represented by the difference $x^n - x^{n-1}$, this method has been shown to significantly improve the convergence rate compared with classical one-step methods for smooth convex optimization problems.

In 2001, *Alvarez* and *Attouch* incorporated an inertial term into the proximal point method and proposed the *inertial proximal point method* for finding the zeros of a maximal monotone operator in Hilbert spaces. The authors introduced the following inertial proximal point algorithm

$$x_{n+1} = J^A_{\lambda_n}(x_n + \theta_n(x_n - x_{n-1})),$$

where $A : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator satisfying $\mathrm{Zer}(A) \neq \emptyset$.

If the sequences of parameters satisfy certain suitable conditions, then the sequence generated by the above algorithm converges weakly to a zero of the operator A. When $\theta_n = 0$, the algorithm reduces to the classical proximal point method of *Martinet* and *Rockafellar*. A distinctive feature of algorithms with inertial terms is that each new iterate depends not only on the previous iterate but also incorporates information from the two preceding iterates. This property has been shown to significantly improve the performance of several earlier algorithms.

Recently, this research direction has been further developed by constructing iterative schemes with multiple inertial terms in order to accelerate the convergence rate. Inertial algorithms and proximal point methods have become essential tools in convex optimization, fixed point theory, and the study of split-type problems, and they play a fundamental role in the development of many modern algorithms.

Based on the above analysis and results, we choose to study the topic **"Some iterative methods for solving the split common solution problem for monotone operator equations with multiple output sets"** for this dissertation, with the aim of contributing to the development of both theoretical results and efficient algorithms for this class of problems.

The objective of this dissertation is to study and develop iterative methods and algorithms for finding solutions to problem (SCSP-MOE) and several related problems. The proposed algorithms are constructed following a self-adaptive approach. More specifically, the objectives are as follows:

- To study the Tikhonov regularization method (implicit iteration) for problem (SCSP-MOE). Based on this regularization approach, an explicit iterative method is proposed for solving this problem.
- To investigate and propose an inertial proximal point method (with regularization) for problem (SCSP-MOE). Moreover, several models combining this method with other techniques, such as the viscosity approximation method, the hybrid projection method, and the shrinking projection method, are introduced in order to construct more efficient algorithms for solving problem (SCSP-MOE).
- To apply the proposed algorithms to several related problems and to construct numerical examples illustrating the obtained theoretical results using MATLAB.

The dissertation is organized into four chapters, together with the introduction, conclusion, a list of symbols, a list of the author's scientific publications, and the bibliography.

Chapter 1: Preliminaries.

Chapter 2: The Tikhonov regularization method for solving problem (SCSP-MOE).

Chapter 3: Inertial proximal point algorithms for solving problem (SCSP-MOE).

Chapter 4: Two projection algorithms with multi-step inertial terms for solving problem (SCSP-MOE).

Chapter 1

Preliminaries

In this chapter, we recall some basic concepts of functional analysis that will be used in the subsequent chapters. These include strong convergence, weak convergence, convex sets, convex functions, the subdifferential of convex functions, several operators in Hilbert spaces, some classes of continuous mappings, metric projections, monotone operators, variational inequalities, and several auxiliary lemmas.

1.1 Basic concepts in Hilbert spaces

1.1.1 Strong convergence and weak convergence

1.1.2 Bounded linear operators and adjoint operators

1.1.3 Convex functions and the subdifferential of convex functions

1.2 Some special classes of mappings

1.2.1 Lipschitz continuous mappings

1.2.2 h-continuous mappings

1.2.3 Metric projections

1.3 Monotone operators

1.4 Variational inequalities in Hilbert spaces

1.5 Some auxiliary lemmas

Chapter 2

Tikhonov Regularization Method for Solving Problem (SCSP-MOE)

In this chapter, we employ the Tikhonov regularization method to construct several algorithms for approximating solutions of problem (SCSP-MOE). In particular, two strongly convergent algorithms based on the implicit iterative method and the explicit iterative method are proposed. Moreover, we investigate the stability of the approximate solutions when the input data are perturbed by noise. The final part of the chapter presents several applications together with numerical examples to illustrate the theoretical results. The results of this chapter are presented based on the content of paper (CT2) listed in the author's publications.

2.1 Implicit Tikhonov Regularization Method

Consider problem (SCSP-MOE) under the following assumptions:

(H1) $\mathcal{H}$ and $\mathcal{H}_i$, $(i = 1, 2, \ldots, N)$ are real Hilbert spaces;
$f \in \mathcal{H}$ and $f_i \in \mathcal{H}_i$, $(i = 1, 2, \ldots, N)$ are given elements.

(H2) $A : \mathcal{H} \to \mathcal{H}$ is a γ-inverse strongly monotone operator on $\mathcal{H}$;
$A_i : \mathcal{H}_i \to \mathcal{H}_i$, $(i = 1, 2, \ldots, N)$ are γ_i-inverse strongly monotone operators on $\mathcal{H}_i$.

(H3) $T_i : \mathcal{H} \to \mathcal{H}_i$, $(i = 1, 2, \ldots, N)$ are bounded linear operators with the corresponding adjoint operators $T_i^* : \mathcal{H}_i \to \mathcal{H}$.

(H4) $\Omega := \Omega_A \cap \left(\bigcap_{i=1}^{N} \Omega_{A_i}\right) \neq \emptyset$, where

$$\Omega_A = \{x \in \mathcal{H} : A(x) = f\}, \qquad \Omega_{A_i} = \{x \in \mathcal{H} : A_i(T_i(x)) = f_i\}.$$

The problem is to find an element $x \in \Omega$.

Based on the classical Tikhonov regularization method presented in the introduction, we propose the following algorithm. The iterative sequence is constructed

by an implicit iterative scheme to solve problem (SCSP-MOE) under assumptions (H1)–(H4) and the following conditions:

(P1) $S : \mathcal{H} \to \mathcal{H}$ is a β-strongly monotone and h-continuous operator on $\mathcal{H}$;

(P2) $\{\alpha_n\}$ is a sequence of positive real numbers satisfying $\lim_{n\to\infty} \alpha_n = 0$.

Algorithm 2.1.1 The implicit Tikhonov regularization method

The sequence $\{x_n\}$ is defined by the equation

$$A(x_n) + \sum_{i=1}^{N} T_i^*\big(A_i(T_i(x_n))\big) + \alpha_n S(x_n) = f + \sum_{i=1}^{N} T_i^*(f_i). \tag{2.1}$$

The following proposition establishes the existence of the iterative sequence $\{x_n\}$ generated by Algorithm 2.1.1.

Proposition 2.1.2. *For each $n \in \mathbb{N}$, equation* (2.1) *has a unique solution x_n.*

The convergence of the iterative sequence generated by Algorithm 2.1.1 is stated in the following theorem.

Theorem 2.1.3. *The sequence $\{x_n\}$ generated by Algorithm 2.1.1 converges strongly to $x^* \in \Omega$, which is the unique solution of the variational inequality*

$$\langle S(x^*),\, q - x^* \rangle \geq 0, \quad \forall q \in \Omega. \tag{2.2}$$

2.2 Explicit Tikhonov Regularization Method

In this section, we construct an explicit iterative method based on the Tikhonov regularization technique to solve problem (SCSP-MOE) under assumptions (H1)–(H4) and the following conditions:

(P1*) $S : \mathcal{H} \to \mathcal{H}$ is a β-strongly monotone and L-Lipschitz continuous operator.

(P2*) $\{\alpha_n\}$ and $\{\theta_n\}$ are sequences of positive real numbers satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \theta_n \alpha_n = \infty$,

(iii) $\lim_{n\to\infty} \dfrac{\theta_n}{\alpha_n} = 0$,

(iv) $\lim_{n\to\infty} \dfrac{|\alpha_{n+1} - \alpha_n|}{\theta_n \alpha_n^2} = 0$.

Algorithm 2.2.1 The explicit Tikhonov regularization method

1: Choose an arbitrary initial point $y_1 \in \mathcal{H}$.

2: **for** $n = 1, 2, \ldots$ **do**

3: Compute

$$F_{\alpha_n}(y_n) = A(y_n) + \sum_{i=1}^{N} T_i^*\big(A_i(T_i(y_n))\big) + \alpha_n S(y_n) - f - \sum_{i=1}^{N} T_i^*(f_i).$$

4: Update

$$y_{n+1} = y_n - \theta_n F_{\alpha_n}(y_n). \tag{2.12}$$

5: **end for**

The strong convergence of the sequence $\{y_n\}$ generated by Alorithm 2.2.1 to a solution of problem (SCSP-MOE) is established in the following theorem.

Theorem 2.2.2. *The sequence $\{y_n\}$ generated by Algorithm 2.2.1 converges strongly to $x^* \in \Omega$, which is the unique solution of the variational inequality* (2.2).

The next part presents the stability of the implicit and explicit Tikhonov regularization methods described above when the input data are perturbed by noise.

2.3 Stability of the Regularization Methods

In this section, we study the stability of the two iterative methods (2.1) and (2.12) under the following noise conditions:

(L1) The elements $f \in \mathcal{H}$ and $f_i \in \mathcal{H}_i$ $(i = 1, 2, \ldots, N)$ are known only approximately. More precisely, f and f_i are replaced by sequences $\{f^{\delta_n}\} \subset \mathcal{H}$ and $\{f_i^{\delta_n^i}\} \subset \mathcal{H}_i$, respectively, satisfying

$$\|f - f^{\delta_n}\| \le \delta_n, \qquad \|f_i - f_i^{\delta_n^i}\| \le \delta_n^i,$$

where $\{\delta_n\}$ và $\{\delta_n^i\}$ are positive real number sequences.

(L2) The operators A and A_i $(i = 1, 2, \ldots, N)$ are given by $A^{h_n} : \mathcal{H} \to \mathcal{H}$ and $A_i^{h_n^i} : \mathcal{H}_i \to \mathcal{H}_i$, respectively, such that A^{h_n} is γ-inverse strongly monotone operator on $\mathcal{H}$ and $A_i^{h_n^i}$ also are γ_i-inverse strongly monotone operator on $\mathcal{H}_i$, respectively, satisfying

$$\|A^{h_n}(x) - A(x)\| \le \varphi(\|x\|)h_n, \qquad \forall x \in \mathcal{H},$$

$$\|A_i^{h_n^i}(x) - A_i(x)\| \le \varphi_i(\|x\|)h_n^i, \qquad \forall x \in \mathcal{H}_i,$$

where $\varphi(t)$ and $\varphi_i(t)$ are bounded functions for all $t \geq 0$, and $\{h_n\}$ and $\{h_n^i\}$ are positive real number sequences .

First, we study the strong convergence of the sequence $\{z_n\}$ defined by

$$A^{h_n}(z_n) + \sum_{i=1}^{N} T_i^*(A_i^{h_n^i}(T_i(z_n))) + \alpha_n S(z_n) = f^{\delta_n} + \sum_{i=1}^{N} T_i^*(f_i^{\delta_n^i}). \tag{2.22}$$

The following theorem presents the stability of the implicit Tikhonov regularization method.

Theorem 2.3.1. *Suppose that conditions (P1), (P2), (L1), and (L2) are satisfied. In addition, if the following condition holds*

$$\lim_{n\to\infty} \frac{\delta_n + h_n + \sum_{i=1}^{N}(\delta_n^i + h_n^i)}{\alpha_n} = 0, \tag{2.23}$$

then the sequence $\{z_n\}$ generated by (2.22) converges strongly to $x^* \in \Omega$, where x^* is the unique solution of the variational inequality (2.2).

Next, we analyze the stability of the explicit Tikhonov regularization method. Specifically, for any $u_1 \in \mathcal{H}$, we consider the sequence $\{u_n\}$ defined by

$$u_{n+1} = u_n - \theta_n \widehat{F}_{\alpha_n}(u_n), \quad n \geq 1, \tag{2.24}$$

where

$$\widehat{F}_{\alpha_n}(u_n) := A^{h_n}(u_n) + \sum_{i=1}^{N} T_i^*\big(A_i^{h_n^i}(T_i(u_n))\big) + \alpha_n S(u_n) - f^{\delta_n} - \sum_{i=1}^{N} T_i^*\big(f_i^{\delta_n^i}\big).$$

Theorem 2.3.2. *Suppose that conditions $(P1^*)$, $(P2^*)$, $(L1)$, $(L2)$ and (2.23) are satisfied. Then the sequence $\{u_n\}$ generated by (2.24) converges strongly to $x^* \in \Omega$, where x^* is the unique solution of the variational inequality (2.2).*

Remark 2.3.3. Theorems 2.3.1 and 2.3.2 show that Algorithm 2.1.1 and Algorithm 2.2.1 are *stable with respect to noise*. This confirms that the proposed algorithms are not only theoretically well-founded but also practically applicable, particularly for modern computational problems where the input data often contain errors.

2.4 Applications and Numerical Examples

In this section, we present several applications of the two algorithms proposed in Chapter 2 for solving the following classes of problems: the split common fixed

point problem with multiple output sets (SCFPP-MOS); the split feasibility problem with multiple output sets (SFP-MOS); the split common null point problem with multiple output sets (SCNPP-MOS); the split common minimization problem with multiple output sets (SMPP-MOS); the signal recovery problem.

In addition, several numerical examples are provided to illustrate the theoretical results and to evaluate the efficiency and convergence behavior of the proposed algorithms. These include an example of problem (SFP-MOS) in an infinite-dimensional Hilbert space, an example of problem (SMPP-MOS) in a finite-dimensional Hilbert space, and an example of a signal recovery problem.

In this summary, we briefly present Example 2.4.7.

Example 2.4.7. We consider problem (SFP-MOS) defined as follows:

$$\text{Find } x \in \widetilde{\Omega} := C \cap \left(\bigcap_{i=1}^{3} T_i^{-1}(Q_i) \right), \tag{2.36}$$

under the following assumptions:

(i) $\mathcal{H} = \mathcal{H}_i = L^2[0,\pi]$ $(i = 1,2,3)$ with the inner product and the induced norm

$$\langle x, y \rangle = \int_0^\pi x(t)\, y(t)\, dt, \quad \|x\| = \left(\int_0^\pi x^2(t)\, dt \right)^{1/2}.$$

The set C and Q_i $(i = 1,2,3)$ are given

$$C = \{\, x \in L^2[0,\pi] : \|x\| \le 2 \,\}, \quad Q_i = \{\, x \in L^2[0,\pi] : \langle a_i, x \rangle \le b_i \,\},$$

where

$$a_i(t) = \sin\big((i+1)t\big), \qquad \forall\, t \in [0,\pi]; \qquad b_i = \frac{1 - \cos(i\pi)}{i(i+1)(i+2)}.$$

(ii) For each $i = 1,2,3$, the operators $T_i : L^2[0,\pi] \longrightarrow L^2[0,\pi]$ are defined by

$$T_i(x)(t) = \frac{x(t)}{i+1}, \qquad \forall\, x = x(t) \in L^2[0,\pi].$$

It is easy to find that $\widetilde{\Omega}$ is a nonempty set because $0 \in \widetilde{\Omega}$. It is not difficult to show that $x_* = 0$ is the unique solution to Problem (2.36).

Applying the explicit Tikhonov regularization method, we obtain

$$z_{n+1} = z_n - \theta_n \Big((I - P_C)(z_n) + \sum_{i=1}^{N} T_i^* \big((I_i - P_{Q_i})(T_i(z_n)) \big) + \alpha_n S(z_n) \Big). \tag{2.31}$$

The data are chosen as follows. Let $S(x) = x$ for all $x = x(t) \in L^2[0, \pi]$, and take the initial point $x_1 = e^t, \quad \forall\, t \in [0, \pi]$. The parameters are selected as

$$\alpha_n = n^{-0.1}, \qquad \theta_n = 0.75\, n^{-0.3}.$$

We use the stopping criterion $\varepsilon_n = \|x_n\| \leq \text{TOL}$, where TOL is a prescribed error tolerance.

In addition, we compare the numerical results of the iteration scheme (2.31) with Algorithm 3.5 proposed by Reich et al. in [52] (Reich, S., Truong, T. M., & Ha, M. T. N. (2020). The split feasibility problem with multiple output sets in Hilbert spaces. *Optimization Letters*, **14**, 2335–2353) and with the algorithm given in Corollary 4.3 of Reich and Tuyen in [54] (Reich, S., & Tuyen, T. M. (2021). Two new self-adaptive algorithms for solving the split common null point problem with multiple output sets in Hilbert spaces. *Journal of Fixed Point Theory and Applications*, **23**(16)).

The data for each algorithm are chosen as follows:

- Algorithm 3.5 in [52]:

$$f(x) = 0.85x, \qquad \alpha_n = n^{-0.1}, \qquad \gamma_n = \frac{2}{3}.$$

- Corollary 4.3 in [54]:

$$f(x) = 0.85x, \qquad \alpha_n = n^{-0.1}, \qquad \beta_{i,n} = 0.75, \qquad \theta_{i,n} = 1, \qquad \rho_{i,n} = 1.$$

The numerical results are reported in Table 2.1.

Table 2.1: Numerical results of (2.31) and comparison with other algorithms

TOL	**Algorithms**	n	ε_n	**Time (s)**
10^{-5}	Algorithms (2.31)	38	9.375×10^{-6}	0.001
	Algorithm 3.5 in [52]	196	9.676×10^{-6}	0.006
	Corollary 4.3 in [54]	71	9.073×10^{-6}	0.029
10^{-6}	Algorithms (2.31)	51	9.144×10^{-7}	0.001
	Algorithm 3.5 in [52]	240	9.627×10^{-7}	0.007
	Corollary 4.3 in [54]	82	9.531×10^{-7}	0.030
10^{-7}	Algorithms (2.31)	65	9.790×10^{-8}	0.001
	Algorithm 3.5 in [52]	285	9.929×10^{-8}	0.008
	Corollary 4.3 in [54]	93	9.897×10^{-8}	0.032

Chapter 3

Inertial Proximal Point Algorithms for Solving the Problem (SCSP-MOE)

In this chapter, we study two iterative algorithms based on the inertial proximal point algorithm (with regularization) for solving problem (SCSP-MOE). The first algorithm is shown to converge weakly to a solution of the problem. We then provide an analysis of its convergence rate. The second algorithm is obtained by combining the weakly convergent algorithm with the viscosity approximation method in order to achieve strong convergence. The final part of the chapter presents several numerical examples to illustrate the proposed algorithms. The results presented in this chapter are based on the main findings of paper (CT1).

3.1 Weak Convergence Algorithm and Convergence Rate

Throughout this chapter, problem (SCSP-MOE) is considered under assumptions (H1)–(H4) stated in Chapter 2.

Based on the inertial proximal point algorithm of Alvarez and Attouch, we construct the following algorithm for solving problem (SCSP-MOE).

Algorithm 3.1.1 Weakly convergent algorithm

1: Choose $\theta \in [0,1)$ and a sequence $\{t_n\}$ satisfying

$$0 \leq t_n \leq \theta, \qquad \forall n \geq 0.$$

2: Choose arbitrary initial points $x_0, x_1 \in \mathcal{H}$.

3: **for** $n = 1, 2, \ldots$ **do**

4: Compute

$$x_{n+1} = J^{A+\sum_{i=1}^{N} T_i^* A_i T_i} \left(x_n + t_n(x_n - x_{n-1}) + f + \sum_{i=1}^{N} T_i^*(f_i) \right). \tag{3.1}$$

5: **end for**

The convergence of the sequence generated by Algorithm 3.1.1 is established in the following theorem.

Theorem 3.1.2. *Assume that the sequence $\{t_n\}$ is chosen such that*

$$\sum_{n=1}^{\infty} t_n \|x_n - x_{n-1}\|^2 < \infty. \tag{C}$$

Then the sequence $\{x_n\}$ generated by Algorithm 3.1.1 converges weakly to a solution of problem (SCSP-MOE).

Remark 3.1.3. (i) By using the inertial proximal point method, we have shown that when condition (C) is satisfied, the sequence $\{x_n\}$ generated by Algorithm 3.1.1 converges weakly to a solution of problem (SCSP-MOE) without requiring any information about the norm of the transfer operator.

(ii) Condition (C) can be guaranteed by choosing an appropriate sequence of parameters. For instance, the sequence $\{t_n\}$ may be chosen as

$$t_n = \frac{1}{(\alpha + n^s)(\|x_n - x_{n-1}\|^2 + \beta)},$$

where $s > 1$ and α, β are positive constants.

Moreover, condition (C) can be automatically satisfied in some special cases, such as the one presented in the following corollary.

Corollary 3.1.4. *Assume that $\{x_n\}$ is the sequence generated by Algorithm 3.1.1. If condition* (C) *is replaced by*

$$\{t_n\} \subset [0, \theta), \quad \theta \in [0, 1/3], \quad \text{and } \{t_n\} \text{ is increasing,} \tag{C'}$$

then we have $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < \infty$, and hence the sequence $\{x_n\}$ converges weakly to some element $p \in \Omega$.

Next, we study the convergence rate of Algorithm 3.1.1 based on the ideas of Chen et al. and Corman et al. First, we establish a preliminary estimate for the sequence $\{x_n\}$ generated by Algorithm 3.1.1 as follows.

Theorem 3.1.5. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1.1 with $x_0 = x_1$, and assume that the sequence $\{t_n\}$ satisfies condition* (C'). *Then for any $p \in \Omega$ and any natural number n, we have*

$$\min_{1 \le k \le n} \|x_{k+1} - z_k\|^2 \le \frac{2(1+\theta^2)(1+\theta-\theta^2)}{(1-\theta)(1-3\theta)n} \|x_0 - p\|^2,$$

where $z_k = x_k + t_k(x_k - x_{k-1})$ for all $k = 1, 2, \ldots, n$.

Remark 3.1.6. In Algorithm 3.1.1, if $x_{n+1} = z_n$, then $x_{n+1} \in \Omega$, that is, x_{n+1} is a solution of problem (SCSP-MOE). Therefore, the stopping criterion

$$\|x_{n+1} - z_n\|^2 < \text{TOL}$$

can be used to terminate the iteration process, where TOL is a prescribed error tolerance.

Next, we establish a sharper error estimate, which confirms that the convergence rate of the sequence actually achieves the order $O(1/\sqrt{n})$.

Theorem 3.1.7. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1.1 with $t_n = \theta < 1/3$ for all $n \geq 1$ and $x_0 = x_1$. Then for any $p \in \Omega$ and every natural number $n \geq 2$, we have*

$$\|x_{n+1} - z_n\|^2 \leq \frac{2(1+7\theta)(1+\theta-\theta^2)(1+\theta^2)}{(1-\theta)(1-2\theta)(1-3\theta)(n-1)}\|x_0 - p\|^2.$$

3.2 Strong Convergence Algorithm

In this section, we introduce an extension of the weakly convergent algorithm (Algorithm 3.1.1) in order to achieve strong convergence.

Algorithm 3.2.1 A strong convergence algorithm

1: Choose $\theta \in [0, 1)$ and a sequence $\{t_n\}$ satisfying $0 \leq t_n \leq \theta, \qquad \forall n \geq 0$.

2: Choose a contraction mapping $\varphi : \mathcal{H} \to \mathcal{H}$ with contraction constant $c \in [0, 1)$.

3: Choose a sequence $\{\alpha_n\} \subset (0, 1)$ satisfying

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

4: Choose arbitrary initial points $x_0, x_1 \in \mathcal{H}$.

5: **for** $n = 1, 2, \ldots$ **do**

6: Compute

$$y_n = x_n + t_n(x_n - x_{n-1}).$$

7: Compute

$$u_n = J^{A+\sum_{i=1}^{N} T_i^* A_i T_i}\left(y_n + f + \sum_{i=1}^{N} T_i^*(f_i)\right).$$

8: Update

$$x_{n+1} = \alpha_n\varphi(x_n) + (1-\alpha_n)u_n. \tag{3.2}$$

9: **end for**

Theorem 3.2.2. *In Algorithm 3.2.1, if the following condition holds*

$$\lim_{n\to\infty} \frac{t_n\|x_n - x_{n-1}\|}{\alpha_n} = 0, \tag{C"}$$

then the sequence $\{x_n\}$ *converges strongly to an element* $p \in \Omega$ *satisfying*

$$p = P_\Omega(\varphi(p)).$$

Remark 3.2.3. (i) By combining the weakly convergent algorithm (Algorithm 3.1.1) with the viscosity approximation method, we establish the strong convergence of the iterative sequence generated by Algorithm 3.2.1 to a solution of problem (SCSP-MOE) without requiring any information about the norm of the transfer operator.

(ii) Algorithms 3.1.1 and 3.2.1 can also be applied to solve several related problems, such as the split common fixed point problem with multiple output sets (SCFPP-MOS), the split feasibility problem with multiple output sets (SFP-MOS), the split common null point problem with multiple output sets (SCNPP-MOS), the split common minimization problem with multiple output sets (SMPP-MOS), and the signal recovery problem.

3.3 Numerical Examples

In this section, we present three numerical examples from Chapter 3: a signal recovery problem, an instance of (SMPP-MOS) in a finite-dimensional Hilbert space, and an instance of (SFP-MOS) in an infinite-dimensional Hilbert space. For brevity, only Example 3.3.1 is outlined here.

Example 3.3.1. Consider the signal recovery problem with the following experimental settings.

The original signal has dimension $n = 100$ with 32 nonzero entries, which are randomly distributed and generated according to a normal distribution.

The main measurement matrix A has size $m \times n$ with $m = 40$. The rows of A are normalized with respect to the Euclidean norm.

For N additional measurement systems, each system consists of a bounded linear operator $T_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$ and a sampling matrix $A_i \in \mathbb{R}^{p_i \times m_i}$, where $m_i \in [50, 100]$ and $p_i \in [20, 50]$ are randomly selected. The rows of the matrices A_i and T_i are normalized with respect to the Euclidean norm and the Frobenius norm, respectively.

To simulate measurement errors, Gaussian noise with variance $\sigma^2 = 0.005$ is added to the observed data.

Algorithm 3.1.1 is then applied with the operators

$$A = \nabla h, \qquad A_i = \nabla h_i, \qquad f = f_i = 0, \quad i = 1, 2, \ldots, N,$$

and the initial values x_0, x_1, and x_2 are randomly generated according to a normal distribution. The sequence $\{x_n\}$ is computed by the iterative formula

$$x_{n+1} = J^{\nabla h + \sum_{i=1}^{N} T_i^* \nabla h_i T_i}(x_n + t_n(x_n - x_{n-1})). \tag{3.30}$$

The algorithm is terminated when $\|x_{n+1} - (x_n + t_n(x_n - x_{n-1}))\| < 10^{-5}$ or when the maximum number of iterations reaches 1000.

The algorithm is tested over 50 independent runs. With the parameter choice $t_n = 0.25$, the corresponding numerical results for different values of N are reported in Table 3.1. The reconstructed signal for $N = 50$ is illustrated in Figure 3.3.

Table 3.1: Numerical results of Algorithm (3.30)

N	Average iterations	Average time (s)	Average error
10	367.26	0.054062	0.059542
20	144.82	0.035797	0.023415
30	96.82	0.031199	0.016185
50	55.76	0.023691	0.009343

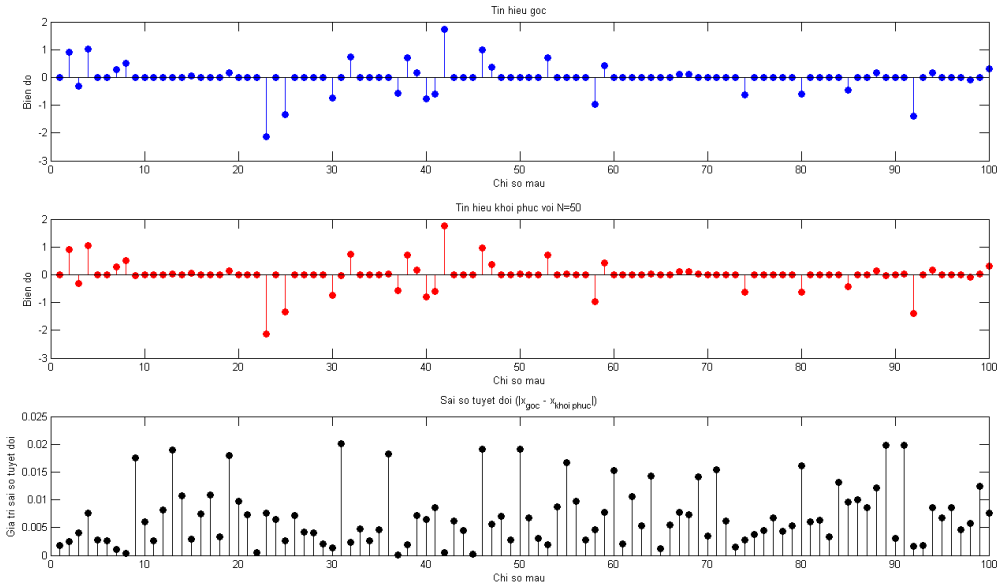


Figure 3.3: Original signal and reconstructed signal obtained by Algorithm (3.30) with $N = 50$

Chapter 4

Two Projection Algorithms with Multi-Step Inertial Terms for Solving Problem (SCSP-MOE)

In this chapter, we present two cyclic projection algorithms for solving problem (SCSP-MOE). These algorithms are developed by combining the proximal point method with multi-step inertial terms and either the hybrid projection technique or the shrinking projection technique. The strong convergence of the two algorithms is established in the corresponding theorems. The final part of this chapter presents two numerical examples to illustrate the theoretical results and to compare the proposed methods with several existing algorithms, thereby demonstrating the effectiveness of the proposed algorithms. The results in this chapter are based on the main findings reported in paper (CT3).

4.1 Hybrid Projection Algorithm

In this chapter, for the convenience of presenting the algorithm and proving the corresponding theorems, we reformulate problem (SCSP-MOE) as follows.

Find an element $p \in \Omega$ under the following assumptions:

(O1) Let $i \in J = \{0, 1, 2, \ldots, N\}$. For each $i \in J$, $\mathcal{H}_i$ is a real Hilbert space and $f_i \in \mathcal{H}_i$ is a given element.

(O2) For each $i \in J$, $A_i : \mathcal{H}_i \to \mathcal{H}_i$ is a γ_i-inverse strongly monotone operator on $\mathcal{H}_i$.

(O3) $T_0 = I_0$ denotes the identity mapping on $\mathcal{H}_0$, and $T_j : \mathcal{H}_0 \to \mathcal{H}_j$ $(j = 1, 2, \ldots, N)$ are bounded linear operators with the corresponding adjoint operators $T_j^* : \mathcal{H}_j \to \mathcal{H}_0$.

(O4) $\Omega = \bigcap_{i \in J} \Omega_{A_i} \neq \emptyset$, where $\Omega_{A_i} = \{p \in \mathcal{H}_0 : A_i(T_i(p)) = f_i\}$ for each $i \in J$.

By combining the cyclic projection method with multi-step inertial terms, we construct the following algorithm for solving problem (SCSP-MOE). In particu-

lar, our algorithm employs an *index control mapping* to enhance the flexibility in determining the order and the selection of mappings at each iteration step.

First, we recall the definition of an index control mapping. A mapping $c : \mathbb{N} \to J$, where $J = \{0, 1, 2, \ldots, N\}$, is called an *index control mapping* if for each $i \in J$, there exists a natural number M_i such that

$$i \in \{c(n), c(n+1), \ldots, c(n+M_i-1)\}, \quad \forall n \in \mathbb{N}.$$

For example, the mapping $c : \mathbb{N} \to J$ defined by

$$c(n) = n \mod (N+1), \quad \forall n \in \mathbb{N},$$

is an index control mapping.

The algorithm is constructed through the following steps.

Algorithm 4.1.1 A hybrid projection algorithm

1: Choose bounded real sequences $\{\ell_{t,n}\}$, $t = 1, 2, \ldots, k$.

2: Choose arbitrary initial points

$$x_{-k}, x_{-(k-1)}, \ldots, x_{-1}, x_0 \in \mathcal{H}_0.$$

3: **for** $n = 0, 1, 2, \ldots$ **do**

4: Compute

$$y_n = x_n + \sum_{t=1}^{k} \ell_{t,n}\big(x_{n+1-t} - x_{n-t}\big).$$

5: Find $z_n \in \mathcal{H}_0$ satisfying

$$T^*_{c(n)}\big(A_{c(n)}(T_{c(n)}(z_n))\big) - T^*_{c(n)}(f_{c(n)}) + z_n = y_n. \tag{4.1}$$

6: Determine

$$C_n = \{a \in \mathcal{H}_0 \,|\, \|z_n - a\| \leq \|y_n - a\|\}, \tag{4.2}$$

$$Q_n = \{b \in \mathcal{H}_0 \,|\, \langle x_0 - x_n,\ b - x_n\rangle \leq 0\}. \tag{4.3}$$

7: Update

$$x_{n+1} = P_{C_n \cap Q_n}(x_0). \tag{4.4}$$

8: **end for**

The strong convergence of the sequence $\{x_n\}$ generated by Algorithm 4.1.1 is established in the following theorem.

Theorem 4.1.4. *The sequence $\{x_n\}$ generated by Algorithm 4.1.1 converges strongly to an element $x_* \in \Omega$ such that $x_* = P_\Omega(x_0)$.*

4.2 Shrinking projection algorithm

In this section, we propose a variant of Algorithm 4.1.1 by using the shrinking projection technique to solve problem (SCSP-MOE).

Algorithm 4.2.1 A shrinking projection algorithm

1: Choose bounded real sequences $\{\ell_{t,n}\}$, $t = 1, 2, \ldots, k$.

2: Choose arbitrary initial points

$$x_{-k}, x_{-(k-1)}, \ldots, x_{-1}, x_0 \in \mathcal{H}_0,$$

and set $\mathcal{W}_{-1} = \mathcal{H}_0$.

3: **for** $n = 0, 1, 2, \ldots$ **do**

4: Compute

$$y_n = x_n + \sum_{t=1}^{k} \ell_{t,n}\big(x_{n+1-t} - x_{n-t}\big).$$

5: Find $z_n \in \mathcal{H}_0$ satisfying

$$T^*_{c(n)}\big(A_{c(n)}(T_{c(n)}(z_n))\big) - T^*_{c(n)}(f_{c(n)}) + z_n = y_n.$$

6: Determine

$$\mathcal{W}_n = \{a \in \mathcal{W}_{n-1} \,|\, \|z_n - a\| \leq \|y_n - a\|\}.$$

7: Update

$$x_{n+1} = P_{\mathcal{W}_n}(x_0).$$

8: **end for**

The following theorem establishes the strong convergence of the sequence generated by the above algorithm to a solution of problem (SCSP-MOE).

Theorem 4.2.2. *The sequence $\{x_n\}$ generated by Algorithm 4.2.1 converges strongly to an element $x_\star \in \Omega$ such that $x_\star = P_\Omega(x_0)$.*

Remark 4.2.3. (i) In some previous works, algorithms for solving problem (SCSP-MOE) require prior knowledge of the inverse strong monotonicity constants of the operators, which are often difficult to determine in practice. In contrast, the two algorithms proposed in Chapter 4 are constructed without relying on these parameters, thereby enhancing their flexibility and broadening their range of applications.

(ii) In earlier algorithms, such as those presented in (CT1), computing x_{n+1} at each iteration requires solving an equation involving the sum of all operators.

By contrast, the two algorithms proposed in this chapter only require computations involving a single pair of operators T_i and A_i at each iteration step, which significantly reduces the computational complexity.

(iii) Algorithms 4.1.1 and 4.2.1 can also be applied to solve several related problems, including the split common fixed point problem with multiple output sets, the split feasibility problem with multiple output sets, and the split common null point problem with multiple output sets.

4.3 Numerical examples

In this section, we present two numerical examples to illustrate the performance of the algorithms proposed in Chapter 4. These include an example of the split minimum point problem with multiple output sets in a finite-dimensional Hilbert space and an example of the split feasibility problem with multiple output sets in an infinite-dimensional Hilbert space. In this summary, we briefly present Example 4.3.1.

Example 4.3.1. Consider the split minimum point problem with multiple output sets:

$$\text{Find } x \in \Omega^* := \bigcap_{i=0}^{3} T_i^{-1}(\operatorname{Argmin} g_i). \tag{4.25}$$

The data are given as follows.

(i) The cost operators $g_0 : \mathbb{R}^5 \to \mathbb{R}$, $g_1 : \mathbb{R}^4 \to \mathbb{R}$, $g_2 : \mathbb{R}^3 \to \mathbb{R}$ and $g_3 : \mathbb{R}^2 \to \mathbb{R}$ are convex and Fréchet differentiable functions defined by

$$\begin{aligned}
g_0(x) &= (x_1 + 2x_2 + x_3 + 2x_4 + x_5 - 1)^2, \qquad \forall\, x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5,\\
g_1(y) &= (y_1 + 2y_2 - y_3 + y_4 - 1)^2, \qquad \forall\, y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4,\\
g_2(z) &= (2z_1 - z_2 + z_3 - 1)^2, \qquad \forall\, z = (z_1, z_2, z_3) \in \mathbb{R}^3,\\
g_3(t) &= (t_1 + t_2 - 1)^2, \qquad \forall\, t = (t_1, t_2) \in \mathbb{R}^2.
\end{aligned}$$

(ii) Let $T_0 = I_0$ be the identity operator on $\mathbb{R}^5$. The transfer mappings $T_1 : \mathbb{R}^5 \to \mathbb{R}^4$, $T_2 : \mathbb{R}^5 \to \mathbb{R}^3$, and $T_3 : \mathbb{R}^5 \to \mathbb{R}^2$ are defined respectively by

$$\begin{aligned}
T_1(x) &= (x_1 + x_2 - 4x_5,\ 2x_3 - x_4 + x_5,\ x_1 + 2x_2 + x_3 - x_4,\ 2x_2 + 3x_5),\\
T_2(x) &= (x_1 - x_2 + x_4,\ -2x_3 + 2x_4 - x_5,\ -2x_1 + 2x_2 - x_3 - x_4),\\
T_3(x) &= (x_1 - x_2 - x_3 + x_4 + x_5,\ -x_1 + x_2 + x_3 - 2x_5),
\end{aligned}$$

for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$.

Then, the solution set of the problem is given by

$$\Omega^* = \{\, x = (5 - 3t,\, -4,\, 2,\, 1 + t,\, t) \in \mathbb{R}^5 : t \in \mathbb{R} \,\}.$$

Applying Algorithm 4.1.1 and Algorithm 4.2.1, where the operators A_i are replaced by $I_i - J^{\partial g_i}$ for $i = 0, 1, 2, 3$, with

$$J^{\partial g_i} = (I_i + \partial g_i)^{-1}, \qquad i = 0, 1, 2, 3,$$

and where I_1, I_2, and I_3 denote the identity operators on $\mathbb{R}^4$, $\mathbb{R}^3$, and $\mathbb{R}^2$, respectively, we obtain two algorithms for solving problem (4.25).

In this example, the initial points are chosen as follows:

$$x_{-3} = (3, 0, 3, 0, 3),\ x_{-2} = (2, 1, 2, 1, 2),\ x_{-1} = (1, 2, 1, 2, 1),\ x_0 = (-4, 0, 4, 0, -4).$$

Moreover, it is easy to verify that $x_* = P_{\Omega^*}(x_0) = (-1,\, -4,\, 2,\, 3,\, 2)$.

We also compare the above two algorithms with Algorithm 2 (denoted by ALGO-H1) and Algorithm 3 (denoted by ALGO-H2) in [35] (Ha, N. S., Tuyen, T. M., & Huyen, P. T. V. (2023). Inertial proximal point algorithm for the split common solution problem of monotone operator equations. *Computational and Applied Mathematics*, **42**, Article 303).

The stopping criterion for all algorithms is chosen as

$$\varepsilon_n = \|x_n - x_*\| < \mathrm{TOL},$$

where TOL denotes a prescribed error tolerance.

The computational results are reported in Tables 4.1, 4.2, and 4.3.

Table 4.1: Numerical results of Algorithm 4.1.1 for Example 4.3.1

Parameter settings	**TOL**	$\boldsymbol{n}$	$\boldsymbol{\varepsilon_n}$	**Time** (s)
$\ell_{1,n} = 1,\ \ell_{2,n} = -3,\ \ell_{3,n} = 2$	10^{-4}	114	9.9000×10^{-5}	0.2854
	10^{-5}	130	9.9781×10^{-6}	0.3011
	10^{-6}	146	9.3456×10^{-7}	0.3232
$\ell_{1,n} = 50,\ \ell_{2,n} = -25,\ \ell_{3,n} = 100$	10^{-4}	134	9.0839×10^{-5}	0.2949
	10^{-5}	150	9.1640×10^{-6}	0.3569
	10^{-6}	166	8.8274×10^{-7}	0.3900
$\ell_{1,n} = 100,\ \ell_{2,n} = -30,\ \ell_{3,n} = -10$	10^{-4}	164	9.3564×10^{-5}	0.3636
	10^{-5}	188	8.0723×10^{-6}	0.4018
	10^{-6}	206	9.9109×10^{-7}	0.4511
$\ell_{1,n} = 1000/n,\ \ell_{2,n} = -50,\ \ell_{3,n} = 25$	10^{-4}	119	9.0477×10^{-5}	0.2712
	10^{-5}	135	9.1169×10^{-6}	0.3115
	10^{-6}	151	8.4995×10^{-7}	0.3393

Table 4.2: Numerical results of Algorithm 4.2.1 for Example 4.3.1

Parameter settings	**TOL**	n	ε_n	**Time** (s)
$\ell_{1,n}=1,\ \ell_{2,n}=-3,\ \ell_{3,n}=2$	10^{-4}	72	9.9920×10^{-5}	0.4212
	10^{-5}	83	7.7280×10^{-6}	0.5535
	10^{-6}	95	6.5143×10^{-7}	0.7336
$\ell_{1,n}=50,\ \ell_{2,n}=-25,\ \ell_{3,n}=100$	10^{-4}	123	8.4315×10^{-5}	0.6706
	10^{-5}	135	7.6039×10^{-6}	0.8289
	10^{-6}	147	7.1771×10^{-7}	0.9956
$\ell_{1,n}=100,\ \ell_{2,n}=-30,\ \ell_{3,n}=-10$	10^{-4}	151	4.7631×10^{-5}	1.1650
	10^{-5}	161	9.8930×10^{-6}	1.3968
	10^{-6}	175	6.4232×10^{-7}	1.6950
$\ell_{1,n}=1000/n,\ \ell_{2,n}=-50,\ \ell_{3,n}=25$	10^{-4}	95	6.7640×10^{-5}	0.4592
	10^{-5}	106	8.0157×10^{-6}	0.5611
	10^{-6}	115	8.7889×10^{-7}	0.6583

Table 4.3: Numerical results of the two algorithms in [35]

TOL	θ_n	**Algorithms**	**n**	ε_n	**Time (s)**
10^{-4}	50	ALGO-H1	1607	9.9735×10^{-5}	5.3989
		ALGO-H2	303	9.9301×10^{-5}	1.6858
	100	ALGO-H1	3202	9.9604×10^{-5}	11.3877
		ALGO-H2	325	8.8377×10^{-5}	2.1084
	$1000/n$	ALGO-H1	263	9.1738×10^{-5}	1.5724
		ALGO-H2	319	9.6868×10^{-5}	2.1917
10^{-5}	50	ALGO-H1	1826	9.9617×10^{-6}	5.9028
		ALGO-H2	362	9.9748×10^{-6}	2.4305
	100	ALGO-H1	3638	9.9611×10^{-6}	13.3407
		ALGO-H2	384	9.8051×10^{-6}	3.0366
	$1000/n$	ALGO-H1	281	9.1894×10^{-6}	1.6091
		ALGO-H2	373	9.0063×10^{-6}	2.9550
10^{-6}	50	ALGO-H1	2040	9.9300×10^{-7}	7.3342
		ALGO-H2	448	9.6430×10^{-7}	3.9373
	100	ALGO-H1	4065	9.9030×10^{-7}	14.9199
		ALGO-H2	474	9.4845×10^{-7}	5.0596
	$1000/n$	ALGO-H1	298	9.2445×10^{-7}	1.6584
		ALGO-H2	431	8.3679×10^{-7}	3.7227

Remark 4.3.2. Tables 4.1–4.3 show that Algorithms 4.1.2 and 4.2.1 achieve the required accuracy while demonstrating superior convergence speed and computational efficiency compared with several previously published methods.

Conclusions

This dissertation studies a new problem in real Hilbert spaces, namely the *split common solution problem for monotone operator equations with multiple output sets* (SCSP-MOE). This model generalizes the split feasibility problem and has potential applications in optimization, equilibrium theory, and signal processing.

The main contributions of the dissertation are summarized as follows:

- Two *Tikhonov regularization algorithms* (implicit and explicit) are proposed for solving problem (SCSP-MOE). The strong convergence and stability of the algorithms are established.
- Two *inertial proximal point algorithms* are developed, including a weakly convergent algorithm and a strongly convergent algorithm, without requiring prior knowledge of the norms of the transfer operators.
- Two *cyclic projection algorithms* based on the hybrid and shrinking projection methods with index control mapping are proposed. These algorithms require computations involving only a single transfer mapping and its corresponding operator at each iteration step, which reduces computational complexity.

Several numerical examples are provided to illustrate the effectiveness of the proposed algorithms and to compare them with existing methods.

Overall, the obtained results contribute to the development of iterative methods for split-type problems in Hilbert spaces and suggest directions for further research.