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**FINITE-TIME STABILITY OF SOME
CLASSES OF FRACTIONAL-ORDER
DIFFERENTIAL SYSTEMS
AND RELATED PROBLEMS**

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SUMMARY OF DOCTORAL THESIS

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Introduction

Fractional calculus investigates derivatives and integrals of non-integer order and traces back to a question posed by L'Hôpital to Leibniz in 1695. Its theoretical development was later advanced by Euler, Lagrange, Liouville, Riemann, Hadamard, and others. Among various definitions, the Riemann–Liouville and Caputo derivatives are the most commonly used. In particular, the Caputo derivative is widely adopted in engineering and physical applications due to its compatibility with classical initial conditions. Fractional-order models often provide more accurate descriptions of real-world dynamics than integer-order models. To capture fast decay effects and finite-memory behavior, the tempered Caputo fractional derivative has been introduced as an extension of the classical Caputo derivative.

Finite-time stability has attracted significant attention due to its practical importance. Originally introduced by G. Kamenkov in 1953 and later developed by P. Dorato in 1961, this concept ensures that the system state remains within a prescribed bound over a finite time interval, in contrast to Lyapunov asymptotic stability, which guarantees convergence only as time approaches infinity.

The performance of dynamical systems is typically assessed via the relationship between input and output signals, reflecting the system's ability to satisfy prescribed operational requirements. From this perspective, several qualitative problems for fractional-order differential systems have been extensively studied, including dissipativity analysis and guaranteed cost control.

The concept of dissipativity, introduced by J. C. Willems in 1972, has become a fundamental tool in the analysis and design of dynamical systems. It generalizes several classical notions, including H_∞ performance and passivity. As a result, dissipativity theory has attracted considerable attention due to its broad applicability in engineering, physics, and control systems, with many significant results reported for various

classes of dynamical systems.

The guaranteed cost control problem was first introduced by S. Chang and T. Peng in 1972. In finite-time control, beyond establishing stability conditions, it is essential to design controllers that guarantee a prescribed performance level. Guaranteed cost control requires not only system stabilization but also that a quadratic cost function be bounded by a given admissible upper limit. Consequently, the integration of guaranteed cost control with finite-time stability plays a crucial role in control system design, enabling an explicit upper bound on a pre-defined performance index. This has motivated extensive research on guaranteed cost control for various classes of dynamical systems.

However, existing results on dissipativity and finite-time guaranteed cost control are largely focused on integer-order systems. For fractional-order systems, particularly those with uncertainties and time delays, corresponding studies remain relatively limited.

For integer-order time-delay systems, analysis methods are typically based on Lyapunov–Krasovskii functionals. However, extending these approaches to uncertain fractional-order systems with delays is considerably more challenging, as it complicates both the construction of Lyapunov-type functionals and the treatment of fractional-order derivatives of quadratic functions, thereby necessitating new analytical techniques.

Motivated by the above considerations, this dissertation focuses on the topic *“Finite-time stability of some classes of fractional-order differential systems and related control problems.”*

In this dissertation, the first class of systems under investigation concerns the extended dissipativity problem for uncertain fractional-order neural networks with disturbances, which are described by the following system of equations

$$\begin{cases} {}^C_0 D_t^\alpha x(t) = -[A + \Delta A(t)]x(t) + [W + \Delta W(t)]g(x(t)) \\ \quad \quad \quad + [D + \Delta D(t)]\omega(t), & t \geq 0, \\ y(t) = C_1 x(t) + C_2 g(x(t)), & t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \quad (1)$$

The results are established using the concept of finite-time extended dissipativity. Fractional-order–dependent sufficient conditions, formu-

lated as linear matrix inequalities, are derived to ensure that uncertain fractional-order neural networks with disturbances satisfy finite-time extended dissipativity over a prescribed interval.

Building on the finite-time extended dissipativity of Caputo fractional-order systems, we further study the finite-time dissipativity of Caputo tempered fractional-order neural networks with disturbances and constant time delays, described by the following system

$$\begin{cases} {}^{TC}D_0^{\alpha,\lambda}x(t) = -[A + \Delta A(t)]x(t) + [B + \Delta B(t)]g(x(t)) \\ \quad + [W + \Delta W(t)]h(x(t - \tau)) + [D + \Delta D(t)]\omega(t), \quad t \geq 0, \\ y(t) = C_1x(t) + C_2x(t - \tau) + C_3g(x(t)) + C_4h(x(t - \tau)) + C_5\omega(t), \\ x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \quad (2)$$

The results are derived by establishing a new lemma for the estimation of quadratic functions under the tempered Caputo fractional derivative. Based on this lemma, sufficient linear matrix inequality (LMI)-based conditions are obtained to ensure the finite-time robust dissipativity of the system.

The subsequent part of this dissertation addresses the finite-time guaranteed cost control of uncertain polytopic fractional-order systems with time-varying delays, described as follows

$$\begin{cases} {}^C_0D^\alpha x(t) = A(\xi)x(t) + D(\xi)x(t - h(t)) + B(\xi)u(t), \quad t > 0, \\ x(\eta) = \varphi(\eta), \quad \forall \eta \in [-h_2, 0]. \end{cases} \quad (3)$$

The objective is to design a state-feedback controller of the form $u^*(t) = Kx(t)$ for system (3) with the cost function

$$J = \frac{1}{\Gamma(\alpha)} \int_0^T (t - s)^{\alpha-1} [x^\top(s)Mx(s) + u^\top(s)Ru(s)] ds, \quad (4)$$

where M and R are symmetric positive definite matrices, such that the resulting closed-loop system

$$\begin{cases} {}^C_0D_t^\alpha x(t) = [A(\xi) + B(\xi)K]x(t) + D(\xi)x(t - h(t)), \\ x(\eta) = \varphi(\eta), \quad \forall \eta \in [-h_2, 0], \end{cases} \quad (5)$$

is finite-time robustly stable and the value of the cost function (4) satisfies $J \leq J^*$, where J^* is a prescribed constant.

The main results provide sufficient conditions for the existence of a finite-time guaranteed cost state-feedback controller for the class of systems described by (3).

The final part of this dissertation focuses on the finite-time guaranteed cost control of uncertain nonlinear Caputo tempered fractional-order systems with time delays, described by the following system

$$\begin{cases} {}^{TC}D_0^{\alpha,\lambda}x(t) = (A + \Delta A(t))x(t) + (D + \Delta D(t))x(t - \tau) \\ \quad + f(t, x(t)) + g(t, x(t - \tau)) + (B + \Delta B(t))u(t), \quad t \geq 0, \\ x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \quad (6)$$

The results provide delay-dependent LMI-based sufficient conditions for designing a finite-time guaranteed cost state-feedback controller for the considered systems..

This dissertation aims to develop analysis and control design methods for uncertain fractional-order systems with delays and disturbances, with particular emphasis on finite-time stability, dissipativity, and guaranteed cost control. Specifically, the main objectives are summarized as follows:

- To investigate finite-time stability and dissipativity of uncertain Caputo fractional-order neural networks with disturbances, as well as finite-time robust dissipativity of uncertain Caputo tempered fractional-order differential systems with disturbances and time delays.
- To study and design finite-time guaranteed cost state-feedback controllers for fractional-order systems, including convex combination systems with time-varying delays and Caputo tempered fractional-order systems with disturbances and time delays.

The main content of the dissertation is organized into three chapters.

- Chapter 1: Mathematical preliminaries.
- Chapter 2: Finite-time dissipativity problem for some classes of uncertain fractional-order neural networks.
- Chapter 3: Finite-time guaranteed cost control problems for several classes of fractional-order control systems.

Chapter 1

Mathematical preliminaries

This chapter presents the fundamental mathematical background that serves as the foundation for the research problems addressed in this dissertation. The contents of this chapter include: basic concepts of fractional calculus, the existence and uniqueness problem for classes of fractional-order differential systems, finite-time stability and finite-time stabilization, dissipativity analysis for dynamical systems, guaranteed cost control problems, and several auxiliary lemmas used in proving the main results of the dissertation.

1.1 Some basic concepts of fractional calculus

1.2 Existence and uniqueness of solutions for fractional-order differential systems

1.3 Finite-time stability and finite-time stabilization

1.4 Dissipativity analysis for dynamical systems

1.5 Guaranteed cost control problems

1.6 Some supporting lemmas

Chapter 2

Finite-time dissipativity problems for fractional-order differential systems

This chapter investigates the finite-time robust extended dissipativity of uncertain Caputo fractional-order neural networks with disturbances, as well as the finite-time robust dissipativity of uncertain Caputo tempered fractional-order systems with disturbances and constant time delay. The results presented are based on the author's published works [CT1, CT4].

2.1 Finite-time dissipativity for uncertain tempered Caputo fractional-order neural networks with time delays

Consider the following class of tempered Caputo fractional-order neural networks with time delay:

$$\begin{cases} {}^{TC}D_0^{\alpha,\lambda}x(t) = -[A + \Delta A(t)]x(t) + [B + \Delta B(t)]g(x(t)) \\ \quad + [W + \Delta W(t)]h(x_\tau(t)) + [D + \Delta D(t)]\omega(t), \quad t \geq 0, \\ y(t) = C_1x(t) + C_2x_\tau(t) + C_3g(x(t)) + C_4h(x_\tau(t)) + C_5\omega(t), \\ x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \quad (2.1)$$

where $x_\tau(t) = x(t - \tau)$, $\alpha \in (0, 1]$ is the fractional order and $\lambda > 0$ is the exponential tempering parameter. The state vector is denoted by $x(t) = (x_1(t), \dots, x_n(t))^T \in \mathbb{R}^n$, and $\omega(t) \in \mathbb{R}^q$ represents the external disturbance input. The dimension n corresponds to the number of neurons in the network. The matrix $A = \text{diag}\{\hat{a}_1, \dots, \hat{a}_n\} \in \mathbb{R}^{n \times n}$ is a positive definite diagonal matrix, while $B, W \in \mathbb{R}^{n \times n}$ and $D \in \mathbb{R}^{n \times q}$ are known constant matrices.

The time-varying uncertain matrices are modeled as

$$\Delta A(t) = E_a F_1(t) H_a, \quad \Delta B(t) = E_b F_2(t) H_b,$$

$$\Delta D(t) = E_d F_3(t) H_d, \quad \Delta W(t) = E_w F_4(t) H_w,$$

where $E_a, E_b, E_d, E_w, H_a, H_b, H_d, H_w$ are known constant matrices with appropriate dimensions. $F_i(t)$ ($i = 1, 2, 3, 4$) are unknown time-varying matrices satisfying the following condition

$$F_i^\top(t) F_i(t) \leq I, \quad \forall t \geq 0, \quad i = 1, 2, 3, 4. \quad (2.2)$$

Here, $\varphi(\cdot) \in C([-\tau, 0], \mathbb{R}^n)$ is the continuous initial condition function with norm $\|\varphi\| = \sup_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|$.

The nonlinear activation functions $g, h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are defined componentwise as follows

$$g(x) = \begin{bmatrix} g_1(x_1(t)) \\ \vdots \\ g_n(x_n(t)) \end{bmatrix}, \quad h(x(t - \tau)) = \begin{bmatrix} h_1(x_1(t - \tau)) \\ \vdots \\ h_n(x_n(t - \tau)) \end{bmatrix},$$

for any $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$.

Assumption 2.1.1. $g_i(\cdot)$ and $h_i(\cdot)$, $i = 1, 2, \dots, n$, are continuous activation functions that satisfy sector conditions, with $g_i(0) = 0$ and $h_i(0) = 0$. Specifically, the following inequalities hold

$$k_i^- \leq \frac{g_i(u) - g_i(v)}{u - v} \leq k_i^+, \quad t_i^- \leq \frac{h_i(u) - h_i(v)}{u - v} \leq t_i^+, \quad (2.3)$$

where $k_i^-, k_i^+, t_i^-, t_i^+$ are known real constants, and $u, v \in \mathbb{R}$ with $u \neq v$.

Assumption 2.1.2. The disturbance input $\omega(\cdot) \in L^2([0, \infty), \mathbb{R}^q)$ satisfies the following condition

$$\exists \bar{\omega} : \omega^\top(t) \omega(t) < \bar{\omega},$$

where $\bar{\omega}$ is a positive scalar.

In the absence of the output vector, the system (2.1) becomes

$$\begin{cases} {}^{TC}D_0^{\alpha, \lambda} x(t) = -[A + \Delta A(t)]x(t) + [B + \Delta B(t)]g(x(t)) \\ \quad + [W + \Delta W(t)]h(x(t - \tau)) + [D + \Delta D(t)]\omega(t), t \geq 0, \\ x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \quad (2.4)$$

Definition 2.1.3. Given positive constants c_1, c_2, T , and $\bar{\omega}$ ($c_1 < c_2$), system (2.4) is said to be robustly finite-time stable with respect to $(c_1, c_2, T, \bar{\omega})$ if the following condition is satisfied:

$$\|\varphi\|^2 \leq c_1 \implies \|x(t)\|^2 < c_2, \quad \forall t \in [0, T],$$

for any disturbance signal $\omega(t) \in \mathbb{R}^q$ satisfying Assumption 2.1.1.

Definition 2.1.4. System (2.1) is said to be dissipative finite-time stable with respect to $(c_1, c_2, T, \bar{\omega})$ if the following two conditions are satisfied:
(i) System (2.4) is robustly finite-time stable with respect to $(c_1, c_2, T, \bar{\omega})$.
(ii) For the zero initial condition $\varphi(t) \equiv 0$ for all $t \in [-\tau, 0]$, there exists a scalar $\gamma > 0$ such that for all $t \in [0, T]$

$$\int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1} \widehat{J}(s) ds \geq \gamma \int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1} \omega^\top(s) \omega(s) ds,$$

where the performance function $\widehat{J}(s)$ is given by

$$\widehat{J}(s) = 2y^\top(s) \Psi_2 \omega(s) + y^\top(s) \Psi_1 y(s) + \omega^\top(s) \Psi_3 \omega(s),$$

with given real matrices Ψ_1 , Ψ_2 , and Ψ_3 . Here, Ψ_1 and Ψ_3 are symmetric matrices, $\Psi_1 \leq 0$, and the disturbance $\omega(t)$ satisfies Assumption 2.1.2.

Theorem 2.1.5. Consider system (2.1) under Assumptions 2.1.1 and 2.2.2. Let c_1, c_2, T ($c_1 < c_2$), and $\bar{\omega}$ be given positive scalars. If there exist positive scalars $r_a, r_b, r_d, r_w, \gamma, \gamma_1, \gamma_2$, a symmetric positive definite matrix P , a symmetric positive definite matrix Q , and diagonal positive definite matrices Λ_g and Λ_h such that the following conditions hold

$$\begin{bmatrix} \Psi_1 & \Psi_2 \\ * & \Psi_3 - Q - \gamma I - \gamma_2 I \end{bmatrix} < 0, \quad (2.5a)$$

$$\begin{bmatrix} \widehat{\Xi}_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & PE_a & PE_b & PE_d & PE_w \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & 0 & 0 & 0 & 0 \\ * & * & \Xi_{33} & \Xi_{34} & \Xi_{35} & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & \Xi_{45} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -r_a I & 0 & 0 & 0 \\ * & * & * & * & * & * & -r_b I & 0 & 0 \\ * & * & * & * & * & * & * & -r_d I & 0 \\ * & * & * & * & * & * & * & * & -r_w I \end{bmatrix} < 0, \quad (2.5b)$$

$$\frac{1}{\lambda_{\min}(P)} (\lambda_1 c_1 + \lambda_2) \leq c_2, \quad (2.5c)$$

$$Q - (\gamma_1 - \gamma_2)I > \gamma_0(1 + e^{\lambda\tau})I, \quad (2.5d)$$

then system (2.1) is dissipative finite-time stable w.r.t. $(c_1, c_2, T, \bar{\omega})$, where

$$\begin{aligned}
K_{1g} &= \text{diag}\{k_1^- k_1^+, k_2^- k_2^+, \dots, k_n^- k_n^+\}, \\
K_{2g} &= \text{diag}\left\{\frac{k_1^- + k_1^+}{2}, \dots, \frac{k_n^- + k_n^+}{2}\right\}, \\
K_{1h} &= \text{diag}\{t_1^- t_1^+, t_2^- t_2^+, \dots, t_n^- t_n^+\}, \\
K_{2h} &= \text{diag}\left\{\frac{t_1^- + t_1^+}{2}, \dots, \frac{t_n^- + t_n^+}{2}\right\}, \\
\Lambda_g &= \text{diag}\{\Lambda_{11}^g, \Lambda_{12}^g, \dots, \Lambda_{1n}^g\}, \Lambda_h = \text{diag}\{\Lambda_{11}^h, \Lambda_{12}^h, \dots, \Lambda_{1n}^h\}, \\
\xi_1 &= E_\alpha(\tau T^\alpha), \xi_2 = e^{\lambda\tau}(E_\alpha(\tau T^\alpha) - 1), \xi_3 = \gamma_1 \bar{\omega} e^{\lambda T} \frac{E_\alpha(\tau T^\alpha) - 1}{\tau}, \\
\lambda_1 &= \lambda_{\max}(P) \xi_1 \sum_{j=0}^{\lfloor T/\tau \rfloor + 1} \xi_2^j, \lambda_2 = \frac{\xi_3}{\xi_1 + \xi_2 - 1} \left(\xi_1 \sum_{j=0}^{\lfloor T/\tau \rfloor + 1} \xi_2^j - 1 \right), \\
\bar{\gamma}_0 &= (\lambda_1 \|\varphi\|^2 + \lambda_2) \tau, x_\tau(t) = x(t - \tau), \\
\gamma_0 &= \frac{\gamma_1 e^{\lambda T} (E_\alpha(\tau T^\alpha) - 1)}{\xi_1 + \xi_2 - 1} \left(\xi_1 \sum_{j=0}^{\lfloor T/\tau \rfloor + 1} \xi_2^j - 1 \right), \\
\hat{\Xi}_{11} &= -PA - A^\top P + r_a H_a^\top H_a - K_{1g} \Lambda_g - C_1^\top \Psi_1 C_1 - \tau P, \\
\Xi_{12} &= -C_1^\top \Psi_1 C_2, \Xi_{13} = PB + K_{2g} \Lambda_g - C_1^\top \Psi_1 C_3, \\
\Xi_{14} &= PW - C_1^\top \Psi_1 C_4, \Xi_{15} = -C_1^\top \Psi_2 - C_1^\top \Psi_1 C_5 + PD, \\
\Xi_{22} &= -K_{1h} \Lambda_h - C_2^\top \Psi_1 C_2 - \tau P, \Xi_{23} = -C_2^\top \Psi_1 C_3, \\
\Xi_{24} &= K_{2h} \Lambda_h - C_2^\top \Psi_1 C_4, \Xi_{25} = -C_2^\top \Psi_2 - C_2^\top \Psi_1 C_5, \\
\Xi_{33} &= r_b H_b^\top H_b - \Lambda_g - C_3^\top \Psi_1 C_3, \Xi_{34} = -C_3^\top \Psi_1 C_4, \\
\Xi_{35} &= -C_3^\top \Psi_2 - C_3^\top \Psi_1 C_5, \\
\Xi_{44} &= r_w H_w^\top H_w - \Lambda_h - C_4^\top \Psi_1 C_4, \Xi_{45} = -C_4^\top \Psi_2 - C_4^\top \Psi_1 C_5, \\
\Xi_{55} &= r_d H_d^\top H_d - C_5^\top \Psi_2 - \Psi_2^\top C_5 - C_5^\top \Psi_1 C_5 \\
&\quad - \Psi_3 + \gamma I - \gamma_1 I + \gamma_2 I + Q, x_\tau(t) = x(t - \tau).
\end{aligned}$$

For $\lambda = 0$, system (2.1) reduces to an uncertain Caputo fractional-

order neural network with time delays.

$$\begin{cases} {}_0^C D_t^\alpha x(t) = -[A + E_a F_a(t) H_a]x(t) + [B + E_b F_b(t) H_b]g(x(t)) \\ \quad + [W + E_w F_w(t) H_w]h(x_\tau(t)) + [D + E_d F_d(t) H_d]\omega(t), \\ y(t) = C_1 x(t) + C_2 x_\tau(t) + C_3 g(x(t)) + C_4 h(x_\tau(t)) + C_5 \omega(t), \\ x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \quad (2.25)$$

From Theorem 2.1.5, the following corollary is obtained:

Corollary 2.1.6. *Assume that positive scalars c_1 , c_2 , T , and $\bar{\omega}$ are given. If there exist positive scalars $r_a, r_b, r_d, r_w, \gamma, \gamma_1, \gamma_2$, symmetric positive definite matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{m \times m}$, and positive definite diagonal matrices Λ_g and Λ_h such that the following conditions are satisfied:*

$$\begin{bmatrix} \Psi_1 & \Psi_2 \\ * & \Psi_3 - Q - \gamma I - \gamma_2 I \end{bmatrix} < 0, \quad (2.26a)$$

$$\begin{bmatrix} \widehat{\Xi}_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & PE_a & PE_b & PE_d & PE_w \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & 0 & 0 & 0 & 0 \\ * & * & \Xi_{33} & \Xi_{34} & \Xi_{35} & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & \Xi_{45} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -r_a I & 0 & 0 & 0 \\ * & * & * & * & * & * & -r_b I & 0 & 0 \\ * & * & * & * & * & * & * & -r_d I & 0 \\ * & * & * & * & * & * & * & * & -r_w I \end{bmatrix} < 0, \quad (2.26b)$$

$$\frac{1}{\lambda_{\min}(P)} (\lambda_1 c_1 + \lambda_2) \leq c_2, \quad (2.26c)$$

$$Q - (\gamma_1 - \gamma_2)I > 2\tilde{\gamma}_0 I, \quad (2.26d)$$

then system (2.25) is dissipative finite-time stable with respect to the parameters $(c_1, c_2, T, \bar{\omega})$.

where

$$\begin{aligned} K_{1g} &= \text{diag} \{k_1^- k_1^+, k_2^- k_2^+, \dots, k_n^- k_n^+\}, \\ K_{2g} &= \text{diag} \left\{ \frac{k_1^- + k_1^+}{2}, \frac{k_2^- + k_2^+}{2}, \dots, \frac{k_n^- + k_n^+}{2} \right\}, \\ K_{1h} &= \text{diag} \{t_1^- t_1^+, t_2^- t_2^+, \dots, t_n^- t_n^+\}, \end{aligned}$$

$$\begin{aligned}
K_{2h} &= \text{diag} \left\{ \frac{t_1^- + t_1^+}{2}, \frac{t_2^- + t_2^+}{2}, \dots, \frac{t_n^- + t_n^+}{2} \right\}, \\
\Lambda_g &= \text{diag}\{\Lambda_{11}^g, \Lambda_{12}^g, \dots, \Lambda_{1n}^g\}, \quad \Lambda_h = \text{diag}\{\Lambda_{11}^h, \Lambda_{12}^h, \dots, \Lambda_{1n}^h\}, \\
\xi_1 &= E_\alpha(\tau T^\alpha), \quad \bar{\xi}_2 = E_\alpha(\tau T^\alpha) - 1, \quad \bar{\xi}_3 = \gamma_1 \bar{\omega} \frac{E_\alpha(\tau T^\alpha) - 1}{\tau}, \\
\lambda_1 &= \lambda_{\max}(P) \xi_1 \sum_{j=0}^{[T/\tau]+1} \bar{\xi}_2^j, \quad \lambda_2 = \frac{\bar{\xi}_3}{\xi_1 + \bar{\xi}_2 - 1} \left(\xi_1 \sum_{j=0}^{[T/\tau]+1} \bar{\xi}_2^j - 1 \right), \\
\bar{\gamma}_0 &= (\lambda_1 \|\varphi\|^2 + \lambda_2) \tau, \quad \tilde{\gamma}_0 = \frac{\gamma_1 (E_\alpha(\tau T^\alpha) - 1)}{\xi_1 + \bar{\xi}_2 - 1} \left(\xi_1 \sum_{j=0}^{[T/\tau]+1} \bar{\xi}_2^j - 1 \right), \\
\hat{\Xi}_{11} &= -PA - A^\top P + r_a H_a^\top H_a - K_{1g} \Lambda_g - C_1^\top \Psi_1 C_1 - \tau P, \\
\Xi_{12} &= -C_1^\top \Psi_1 C_2, \quad \Xi_{13} = PB + K_{2g} \Lambda_g - C_1^\top \Psi_1 C_3, \quad \Xi_{14} = PW - C_1^\top \Psi_1 C_4, \\
\Xi_{15} &= -C_1^\top \Psi_2 - C_1^\top \Psi_1 C_5 + PD, \quad \Xi_{22} = -K_{1h} \Lambda_h - C_2^\top \Psi_1 C_2 - \tau P, \\
\Xi_{23} &= -C_2^\top \Psi_1 C_3, \quad \Xi_{24} = K_{2h} \Lambda_h - C_2^\top \Psi_1 C_4, \quad \Xi_{25} = -C_2^\top \Psi_2 - C_2^\top \Psi_1 C_5, \\
\Xi_{33} &= r_b H_b^\top H_b - \Lambda_g - C_3^\top \Psi_1 C_3, \quad \Xi_{34} = -C_3^\top \Psi_1 C_4, \\
\Xi_{35} &= -C_3^\top \Psi_2 - C_3^\top \Psi_1 C_5, \quad \Xi_{44} = r_w H_w^\top H_w - \Lambda_h - C_4^\top \Psi_1 C_4, \\
\Xi_{45} &= -C_4^\top \Psi_2 - C_4^\top \Psi_1 C_5, \\
\Xi_{55} &= r_d H_d^\top H_d - C_5^\top \Psi_2 - \Psi_2^\top C_5 - C_5^\top \Psi_1 C_5 - \Psi_3 + \gamma I - \gamma_1 I + \gamma_2 I + Q.
\end{aligned}$$

2.2 Finite-time extended dissipativity for uncertain Caputo fractional-order neural networks with disturbances

This research studies the following fractional-order neural networks with uncertainties:

$$\begin{cases} {}_0^C D_t^\alpha x(t) = -[A + \Delta A(t)]x(t) + [W + \Delta W(t)]g(x(t)) \\ \quad \quad \quad + [D + \Delta D(t)]\omega(t), \quad t \geq 0, \\ y(t) = C_1 x(t) + C_2 g(x(t)), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \quad (2.28)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^\top \in \mathbb{R}^n$ denotes the state vector, $y(t) \in \mathbb{R}^m$ is the output vector, and $\omega(\cdot) \in L^2([0, \infty), \mathbb{R}^q)$ denotes the

input disturbance. Moreover,

$$g(x(t)) = (g_1(x_1(t)), \dots, g_n(x_n(t)))^\top$$

represents the activation functions. Here, $A = \text{diag}\{a_1, \dots, a_n\}$ is a positive definite diagonal matrix, and W, D, C_1, C_2 are known constant matrices of appropriate dimensions.

Assumption 2.2.1. *The time-varying parameter uncertainties matrices $\Delta A(t)$, $\Delta D(t)$, and $\Delta W(t)$ satisfy the following conditions:*

$$\Delta A(t) = E_1 F_1(t) H_1, \quad \Delta W(t) = E_2 F_2(t) H_2, \quad \Delta D(t) = E_3 F_3(t) H_3.$$

Here, E_1, E_2, E_3, H_1, H_2 , and H_3 are known real constant matrices of appropriate dimensions, $F_i(t) (i = 1, 2, 3)$ are unknown time-varying matrices satisfying $F_i^\top(t) F_i(t) \leq I, \quad \forall t \geq 0, (i = 1, 2, 3)$.

Assumption 2.2.2. $g_i(\cdot), i = 1, 2, \dots, n$, are nonlinear activation functions that are continuous and bounded with $g_i(0) = 0$, and satisfy

$$k_i^- \leq \frac{g_i(u) - g_i(v)}{u - v} \leq k_i^+, \quad (2.29)$$

where k_i^- and k_i^+ are known real constants, and $u, v \in \mathbb{R}$ with $u \neq v$.

In the absence of the output vector $y(t)$, the system (2.28) becomes

$$\begin{cases} {}^C_0 D_t^\alpha x(t) = -[A + \Delta A(t)]x(t) + [W + \Delta W(t)]g(x(t)) \\ \quad + [D + \Delta D(t)]\omega(t), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \quad (2.30)$$

Definition 2.2.3. For given a matrix $R \in \mathbb{S}_n^+$, positive scalars T, c_1, c_2 ($c_1 < c_2$), and the disturbance input $\omega(t)$ satisfying Assumption 2.1.2, system (2.30) is said to be robustly finite-time bounded with respect to (w.r.t.) $(c_1, c_2, T, \bar{\omega}, R)$ if $x_0^\top R x_0 \leq c_1 \Rightarrow x^\top(t) R x(t) < c_2, \forall t \in [0, T]$.

Definition 2.2.4. Given matrices Ψ_1, Ψ_2, Ψ_3 , and Ψ_4 satisfying Assumption ??, system (2.28) is said to be robustly finite-time extended dissipative w.r.t. $(c_1, c_2, T, \bar{\omega}, R)$ if two conditions hold:

1. System (2.30) is robustly finite-time bounded w.r.t. $(c_1, c_2, T, \bar{\omega}, R)$.
2. Under the zero initial state, the following condition is met

$${}_0 I_t^\alpha \hat{J}(t) - y^\top(t) \Psi_4 y(t) \geq 0, \quad \forall t \in [0, T], \quad (2.31)$$

where $J(t) = y^\top(t)\Psi_1 y(t) + 2y^\top(t)\Psi_2 \omega(t) + \omega^\top(t)\Psi_3 \omega(t)$, with Ψ_1, Ψ_3 , and Ψ_4 are symmetric real matrices of appropriate dimensions satisfying $\Psi_1 \leq 0$, $\Psi_3 \geq 0$, and $\Psi_4 \geq 0$, respectively, and Ψ_2 is a real matrix of appropriate dimensions.

Theorem 2.2.6. *Given positive scalars $c_1, c_2, \bar{\omega}, T$, and a matrix $R \in \mathbb{S}_n^+$. Under Assumptions 2.2.1–2.1.2, system (2.28) is robustly finite-time extended dissipative w.r.t. $(c_1, c_2, T, R, \bar{\omega})$ if there exist three positive scalars $\epsilon_1, \epsilon_2, \epsilon_3$, two matrices $Q, X \in \mathbb{S}_n^+$, and a diagonal positive matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that the following conditions hold*

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & QE_1 & QE_2 & QE_3 \\ * & \Omega_{22} & -C_2^\top \Psi_2 & 0 & 0 & 0 \\ * & * & \Omega_{33} & 0 & 0 & 0 \\ * & * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & * & -\epsilon_2 I & 0 \\ * & * & * & * & * & -\epsilon_3 I \end{bmatrix} < 0, \quad (2.32a)$$

$$\lambda_2 c_1 + \frac{\lambda_{\max}(\Psi_3) \bar{\omega}}{\Gamma(\alpha + 1)} T^\alpha < \lambda_1 c_2, \quad (2.32b)$$

$$\begin{bmatrix} C_1^\top \Psi_4 C_1 & C_1^\top \Psi_4 C_2 \\ * & C_2^\top \Psi_4 C_2 \end{bmatrix} < \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}, \quad (2.32c)$$

$$X + K^2 < Q, \quad (2.32d)$$

where

$$K = \text{diag} \{k_1^+, k_2^+, \dots, k_n^+\}, \quad K_1 = \text{diag} \{k_1^- k_1^+, k_2^- k_2^+, \dots, k_n^- k_n^+\},$$

$$K_2 = \text{diag} \left\{ \frac{k_1^- + k_1^+}{2}, \frac{k_2^- + k_2^+}{2}, \dots, \frac{k_n^- + k_n^+}{2} \right\},$$

$$\bar{Q} = R^{-\frac{1}{2}} Q R^{-\frac{1}{2}}, \quad \lambda_1 = \lambda_{\min}(\bar{Q}), \quad \lambda_2 = \lambda_{\max}(\bar{Q}),$$

$$\Omega_{11} = -QA - A^\top Q + \epsilon_1 h_1^\top h_1 - C_1^\top \Psi_1 C_1 - K_1 \Lambda,$$

$$\Omega_{22} = \epsilon_2 h_2^\top h_2 - C_2^\top \Psi_1 C_2 - \Lambda,$$

$$\Omega_{33} = \epsilon_3 H_3^\top H_3 - \Psi_3, \quad \Omega_{12} = QW - C_1^\top \Psi_1 C_2 + K_2 \Lambda, \quad \Omega_{13} = QD - C_1^\top \Psi_2.$$

By setting $\Psi_1 = 0$, $\Psi_2 = I$, $\Psi_3 = \gamma I$, and $\Psi_4 = 0$, where $\gamma > 0$, the passivity criteria for system (2.28) follow directly from Theorem 2.2.6:

Corollary 2.2.9. *Given positive scalars $c_1, c_2, \bar{\omega}, T$ and a matrix $R \in \mathbb{S}_n^+$. Under Assumptions 2.2.1–2.1.2, system (2.28) is robustly finite-time*

passive w.r.t. $(c_1, c_2, T, R, \bar{\omega})$ if there exist positive scalars $\epsilon_1, \epsilon_2, \epsilon_3, \gamma$, matrices $Q, X \in \mathbb{S}_n^+$, and a diagonal positive matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that the following conditions hold

$$\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & QE_1 & QE_2 & QE_3 \\ * & \mathcal{M}_{22} & -C_2^\top & 0 & 0 & 0 \\ * & * & \mathcal{M}_{33} & 0 & 0 & 0 \\ * & * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & * & -\epsilon_2 I & 0 \\ * & * & * & * & * & -\epsilon_3 I \end{bmatrix} < 0, \quad (2.52a)$$

$$\lambda_2 c_1 + \frac{\gamma \bar{\omega}}{\Gamma(\alpha + 1)} T^\alpha < \lambda_1 c_2, \quad (2.52b)$$

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} > 0, \quad (2.52c)$$

$$X + K^2 < Q, \quad (2.52d)$$

where

$$K = \text{diag}\{k_1^+, k_2^+, \dots, k_n^+\}, \quad K_1 = \text{diag}\{k_1^- k_1^+, k_2^- k_2^+, \dots, k_n^- k_n^+\},$$

$$K_2 = \text{diag}\left\{\frac{k_1^- + k_1^+}{2}, \frac{k_2^- + k_2^+}{2}, \dots, \frac{k_n^- + k_n^+}{2}\right\},$$

$$\bar{Q} = R^{-1/2} Q R^{-1/2}, \quad \lambda_1 = \lambda_{\min}(\bar{Q}), \quad \lambda_2 = \lambda_{\max}(\bar{Q}),$$

$$\mathcal{M}_{11} = -QA - A^\top Q + \epsilon_1 H_1^\top H_1 - K_1 \Lambda,$$

$$\mathcal{M}_{22} = \epsilon_2 H_2^\top H_2 - \Lambda, \quad \mathcal{M}_{33} = \epsilon_3 H_3^\top H_3 - \gamma I,$$

$$\mathcal{M}_{12} = QW + K_2 \Lambda, \quad \mathcal{M}_{13} = QD - C_1^\top.$$

We now consider a special case of system (2.28), when $\Delta A(t) = 0$, $\Delta W(t) = 0$, and $\Delta D(t) = 0$, the system reduces to the following fractional-order neural networks

$$\begin{cases} {}^C_0 D_t^\alpha x(t) = -Ax(t) + Wg(x(t)) + D\omega(t), \quad t \geq 0, \\ y(t) = C_1 x(t) + C_2 g(x(t)), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \quad (2.53)$$

According to Theorem 2.2.6, the following corollary is obtained.

Corollary 2.2.10. *Given positive scalars $c_1, c_2, \bar{\omega}, T$ with $c_1 < c_2$ and a matrix $R \in \mathbb{S}_n^+$. Under Assumptions 2.1.2 and 2.2.2, system (2.53)*

is robustly finite-time extended dissipative w.r.t. $(c_1, c_2, T, R, \bar{\omega})$ if there exist two matrices $Q, X \in \mathbb{S}_n^+$ and a diagonal positive matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that the following conditions hold

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & QD - C_1^\top \Psi_2 \\ * & -C_2^\top \Psi_1 C_2 - \Lambda & -C_2^\top \Psi_2 \\ * & * & -\Psi_3 \end{bmatrix} < 0, \quad (2.54a)$$

$$\lambda_2 c_1 + \frac{\lambda_{\max}(\Psi_3) \bar{\omega}}{\Gamma(\alpha + 1)} T^\alpha < \lambda_1 c_2, \quad (2.54b)$$

$$\begin{bmatrix} C_1^\top \Psi_4 C_1 & C_1^\top \Psi_4 C_2 \\ * & C_2^\top \Psi_4 C_2 \end{bmatrix} < \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}, \quad (2.54c)$$

$$X + K^2 < Q, \quad (2.54d)$$

where

$$\Phi_{11} = -QA - A^\top Q - C_1^\top \Psi_1 C_1 - K_1 \Lambda, \quad \Phi_{12} = QW - C_1^\top \Psi_1 C_2 + K_2 \Lambda.$$

2.3 Conclusion of Chapter 2

This chapter investigated finite-time dissipativity for several classes of fractional-order neural networks, including tempered Caputo systems with disturbances and delays, as well as uncertain Caputo systems. The main results are summarized as follows

- For tempered Caputo fractional-order neural networks with external disturbances and constant delays, sufficient conditions ensuring finite-time dissipativity were established in terms of linear matrix inequalities by employing the Lyapunov functional method.
- For uncertain Caputo fractional-order neural networks, sufficient conditions guaranteeing finite-time extended dissipativity were derived in terms of linear matrix inequalities based on the Lyapunov functional approach.
- Numerical examples and simulations were provided to illustrate the feasibility of the proposed sufficient conditions. The simulation results demonstrated that the obtained theoretical criteria are effective and can be verified numerically.

Chapter 3

Finite-time guaranteed cost control problems for fractional-order control systems

This chapter presents research results related to the problem of finite-time guaranteed cost control for classes of uncertain Caputo fractional-order systems with delays. Specifically, the chapter focuses on the design of guaranteed cost controllers and on deriving sufficient conditions for the existence of finite-time guaranteed cost state-feedback controllers for uncertain polytopic Caputo fractional-order systems with time-varying delays, formulated in terms of linear matrix inequalities (LMIs), as well as for uncertain tempered Caputo fractional-order differential systems with time delays. The content of this chapter is developed based on two published papers [CT2, CT3] in the author's list of scientific publications.

3.1 Finite-time guaranteed cost control of uncertain polytopic fractional-order control systems with time-varying delays

Consider the uncertain polytopic system as follows

$$\begin{cases} {}^C_0 D^\alpha x(t) = A(\xi)x(t) + D(\xi)x(t - h(t)) + B(\xi)u(t), & t > 0, \\ x(\eta) = \varphi(\eta), & \forall \eta \in [-h_2, 0], \end{cases} \quad (3.1)$$

where $0 < \alpha < 1$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control, the system matrices $\{A(\xi), D(\xi), B(\xi)\}$ are uncertain but belong to a polytope Ω given by

$$\Omega := \left\{ [A, B, D](\xi) = \sum_{i=1}^s \xi_i [A_i, B_i, D_i] \mid \sum_{i=1}^s \xi_i = 1, \xi_i > 0, i \in \{1, \dots, s\} \right\},$$

where $A_i, D_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}, i \in \overline{1, s}, s \geq 1$, are given constant matrices; the time delay function $h(t)$ satisfies $0 < h_1 \leq h(t) \leq h_2, \forall t \geq 0$; the initial function $\varphi(t) \in C([-h_2, 0], \mathbb{R}^n)$ with the norm

$$\varphi(t) \in C([-h_2, 0], \mathbb{R}^n) \text{ with } \|\varphi\| = \sup_{s \in [-h_2, 0]} \|\varphi(s)\|.$$

For a given positive scalar $T > 0$, associated with system (3.1) is the following quadratic cost function:

$$J = \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} [x^\top(s)Mx(s) + u^\top(s)Ru(s)] ds, \quad (3.2)$$

where M and R are symmetric positive definite matrices.

In the absence of the control $u(t)$, the system (3.1) becomes

$$\begin{cases} {}^C_0 D_t^\alpha x(t) = A(\xi)x(t) + D(\xi)x(t-h(t)), \\ x(\eta) = \varphi(\eta), \quad \forall \eta \in [-h_2, 0]. \end{cases} \quad (3.3)$$

Definition 3.1.1. *Given $c_2 \geq c_1 > 0$ and $T > 0$, system (3.3) is robustly finite-time stable w.r.t. (c_1, c_2, T) if the following condition holds:*

$$\|\varphi\|^2 \leq c_1 \Rightarrow \|x(t)\|^2 \leq c_2, \quad \forall t \in [0, T].$$

Definition 3.1.2. *Given $c_2 \geq c_1 > 0$ and $T > 0$, system (3.1) is robustly finite-time stabilizable w.r.t. (c_1, c_2, T) if there exists a state-feedback controller of the form $u(t) = Kx(t)$ such that the closed-loop system*

$$\begin{cases} {}^C_0 D_t^\alpha x(t) = [A(\xi) + B(\xi)K]x(t) + D(\xi)x(t-h(t)), \\ x(\eta) = \varphi(\eta), \quad \forall \eta \in [-h_2, 0], \end{cases} \quad (3.4)$$

is robustly finite-time stable w.r.t. (c_1, c_2, T) .

Definition 3.1.3. *For system (3.1) and the cost function (3.2), if there exist a state-feedback controller $u^*(t)$ and a positive constant J^* such that the closed-loop system (3.4) is finite-time robustly stable with respect to the parameter set (c_1, c_2, T) and the value of the cost function satisfies $J \leq J^*$, then J^* is called the guaranteed cost.*

Our objective is to design a state-feedback controller $u^*(t) = Kx(t)$ for system (3.1) and the cost function (3.2) such that the closed-loop system (3.4) is finite-time robustly stable and the value of the cost function (3.2) satisfies $J \leq J^*$, where J^* is a given constant.

First, to formulate the LMI conditions in the controller design procedure, we consider the indices $i, j = 1, 2, \dots, s$ with $i \neq j$ and define the matrix:

$$\mathbb{S} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix},$$

where S is an unknown symmetric positive definite matrix. The following expressions are used in the construction of the LMI conditions:

$$W_{ii} = A_i P + B_i Y + (A_i P + B_i Y)^\top - h_2 P + S,$$

$$W_{jj} = A_i P + B_i Y + A_j P + B_j Y + (A_i P + B_i Y + A_j P + B_j Y)^\top - 2h_2 P - \frac{2S}{s-1},$$

$$q = E_\alpha(h_2 T^\alpha) \sum_{j=0}^{[T/h_1]+1} (E_\alpha(h_2 T^\alpha) - 1)^j,$$

$$\beta = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}, \quad \beta_0 = \frac{\frac{2h_2 q T^\alpha}{\Gamma(\alpha+1)} + 1}{\lambda_{\min}(P)}.$$

Theorem 3.1.4. *Let $c_2 > c_1 > 0$ and $T > 0$. Consider system (3.1) with the cost function (3.2). If there exist symmetric positive definite matrices P and S , and a free weighting matrix Y such that the following conditions hold*

$$\begin{bmatrix} W_{ii} & D_i P & P M & Y^\top R \\ * & -h_2 P & 0 & 0 \\ * & * & -M & 0 \\ * & * & * & -R \end{bmatrix} < 0, \quad \forall i \in \overline{1, s}, \quad (3.5a)$$

$$\begin{bmatrix} W_{jj} & D_i P + D_j P & P M & Y^\top R \\ * & -2h_2 P & 0 & 0 \\ * & * & -\frac{1}{2}M & 0 \\ * & * & * & -\frac{1}{2}R \end{bmatrix} < 0, \quad \forall i \neq j, i, j \in \overline{1, s}, \quad (3.5b)$$

$$E_\alpha(h_2 T^\alpha) \sum_{j=0}^{[T/h_1]+1} (E_\alpha(h_2 T^\alpha) - 1)^j \cdot \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \leq \frac{c_2}{c_1}, \quad (3.5c)$$

then $u(t) = Y P^{-1} x(t)$ is the guaranteed control law and the guaranteed

cost value is

$$J \leq J^* = \frac{\frac{2h_2qT^\alpha}{\Gamma(\alpha+1)} + 1}{\lambda_{\min}(P)} \|\varphi\|^2.$$

When $s = 2$, $A_i = A$, $D_i = D$, $B_i = B$, $\forall i \in \{1, 2\}$, the system (3.1) is reduced to the following linear Caputo fractional-order system with an interval time-varying delay

$$\begin{cases} D^\alpha x(t) = Ax(t) + Dx(t - h(t)) + Bu(t), \\ x(t) = \varphi(t), \end{cases} \quad \forall t \in [0, T], \quad (3.27)$$

where $0 < \alpha < 1$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control, $A, D \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are given constant matrices; the delay $h(t)$ satisfies $0 < h_1 \leq h(t) \leq h_2$, $\forall t \geq 0$; the initial function $\varphi(t) \in C([-h_2, 0], \mathbb{R}^n)$ with the norm $\|\varphi\| = \sup_{\theta \in [-h_2, 0]} \|\varphi(\theta)\|$.

From Theorem 3.1.4, the following corollary is obtained:

Corollary 3.1.5. *Given positive numbers c_1, c_2, T ($c_2 \geq c_1$). Consider system (3.27) with the cost function (3.2). If there exists symmetric positive definite matrices $P, S \in \mathbb{R}^{n \times n}$, and a free weighting matrix Y such that the following conditions hold*

$$\begin{bmatrix} W & DP & PM & Y^\top R \\ * & -h_2P & 0 & 0 \\ * & * & -M & 0 \\ * & * & * & -R \end{bmatrix} < 0, \quad (3.28a)$$

$$\begin{bmatrix} W_1 & 2DP & 2PM & 2Y^\top R \\ * & -2h_2P & 0 & 0 \\ * & * & -\frac{1}{2}M & 0 \\ * & * & * & -\frac{1}{2}R \end{bmatrix} < 0, \quad (3.28b)$$

$$E_\alpha(h_2T^\alpha) \sum_{j=0}^{\lceil T/h_1 \rceil + 1} (E_\alpha(h_2T^\alpha) - 1)^j \cdot \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \leq \frac{c_2}{c_1}, \quad (3.28c)$$

then $u(t) = YP^{-1}x(t)$ is the guaranteed control law and the guaranteed

cost value is

$$J^* = \frac{\frac{2h_2qT^\alpha}{\Gamma(\alpha+1)} + 1}{\lambda_{\min}(P)} \|\varphi\|^2,$$

where

$$\begin{aligned} W &= [AP + BY] + [AP + BY]^\top - h_2P + S, \\ W_1 &= 2[AP + BY] + 2[AP + BY]^\top - 2h_2P - 2S, \\ q &= E_\alpha(h_2T^\alpha) \sum_{j=0}^{\lceil T/h_1 \rceil + 1} (E_\alpha(h_2T^\alpha) - 1)^j, \\ \beta &= \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}, \quad \beta_0 = \frac{\left(\frac{2h_2qT^\alpha}{\Gamma(\alpha+1)} + 1\right)}{\lambda_{\min}(P)}. \end{aligned}$$

3.2 Finite-time guaranteed cost control for nonlinear tempered fractional-order systems with delays

Consider an uncertain tempered Caputo fractional-order system with time delays and nonlinear perturbations

$$\begin{cases} {}^{TC}D_0^{\alpha,\lambda}x(t) = (A + \Delta A(t))x(t) + (D + \Delta D(t))x(t - \tau) \\ \quad + f(t, x(t)) + g(t, x(t - \tau)) + (B + \Delta B(t))u(t), \quad t \geq 0, \\ x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \quad (3.30)$$

where $\lambda \geq 0$, $\alpha \in (0, 1]$ denotes the fractional order of the system; $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^m$ is the control input; The initial condition is given by a function $\varphi(\cdot) \in C([-\tau, 0], \mathbb{R}^n)$ with norm $\|\varphi\| = \sup_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|$. The matrices A , B , and D are given constant real matrices with appropriate dimensions; $\Delta A(t)$, $\Delta B(t)$, and $\Delta D(t)$ are time-varying uncertain matrices; $f(\cdot)$ and $g(\cdot)$ are nonlinear activation functions.

Assumption 3.2.1. *The time-varying uncertain matrices are modeled as: $\Delta A(t)$, $\Delta B(t)$, $\Delta W(t)$ and $\Delta B(t)$ are assumed to take the following forms:*

$$\Delta A(t) = E_a F_1(t) H_a, \quad \Delta D(t) = E_d F_2(t) H_d, \quad \Delta B(t) = E_b F_3(t) H_b,$$

where E_a, E_d, E_d, H_a, H_b , and H_d are known constant real matrices with appropriate dimensions, and $F_i(t)$ ($i = 1, 2, 3$) are time-varying matrices satisfying

$$F_i^\top(t)F_i(t) \leq I, \quad \forall t \geq 0, \quad i = 1, 2, 3.$$

Assumption 3.2.2. Consider the functions $f, g : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, which satisfy the following quadratic boundedness conditions:

$$\begin{aligned} f^\top(t, x)f(t, x) &\leq x^\top Qx, \\ g^\top(t, x)g(t, x) &\leq x^\top Gx, \end{aligned} \quad (3.31)$$

for all $(t, x) \in [0, +\infty) \times \mathbb{R}^n$, where Q and G are given symmetric positive definite matrices.

For a given positive scalar $T > 0$, the cost function associated with system (3.30) is defined as:

$$J(t) = \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1} [x^\top(s)Mx(s) + u^\top(s)Ru(s)] ds, \quad (3.32)$$

for all $t \in [0, T]$, where M and R are given symmetric positive definite matrices.

When the control input $u(t)$ is absent, system (3.30) reduces to

$$\begin{cases} {}^{TC}D_0^{\alpha, \lambda} x(t) = (A + \Delta A(t))x(t) + (D + \Delta D(t))x(t - \tau) \\ \quad + f(t, x(t)) + g(t, x(t - \tau)), \quad t \geq 0, \\ x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \quad (3.33)$$

Definition 3.2.3. Consider positive constants c_1, c_2 , and T . The unforced system (3.33) (i.e., when $u(t) = 0$) is defined as robustly finite-time stable with respect to (c_1, c_2, T) if the following condition holds:

$$\|\varphi\|^2 < c_1 \implies \|x(t)\|^2 < c_2, \quad \forall t \in [0, T].$$

Definition 3.2.4. Consider positive constants c_1, c_2 with $c_2 > c_1$, and a finite time horizon $T > 0$. The system (3.30) is said to be robustly finite-time stabilizable with respect to the triple (c_1, c_2, T) if there exists a state feedback control law of the form $u(t) = Kx(t)$ such that the resulting closed-loop system

$$\begin{cases} {}^{TC}D_0^{\alpha, \lambda} x(t) = (A + \Delta A(t) + BK + \Delta B(t))x(t) \\ \quad + (D + \Delta D(t))x(t - \tau) + f(t, x(t)) + g(t, x(t - \tau)), \quad t \geq 0, \\ x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \quad (3.34)$$

is robustly finite-time stable with respect to (c_1, c_2, T) .

The guaranteed cost controller $u(t) = Kx(t)$ is designed for system (3.30) such that the closed-loop system (3.34) is robustly finite-time stable with respect to (c_1, c_2, T) . Additionally, the cost function (3.32) satisfies

$$J(t) \leq J^*, \quad \forall t \in [0, T],$$

where J^* is a predetermined constant called the guaranteed cost value of the system.

Theorem 3.2.5. *Consider positive constants c_1, c_2 with $c_2 > c_1$, and a finite time horizon $T > 0$. Suppose there exist positive scalars $\epsilon_1, \epsilon_2, \epsilon_3$, a symmetric positive-definite matrix $P \in \mathbb{R}^{n \times n}$, and a free-weighting matrix $Y \in \mathbb{R}^{m \times n}$ such that the following conditions hold*

$$\begin{bmatrix} \Theta_{11} & DP & I & I & PQ & PM & Y^\top R & PH_a^\top & Y^\top H_b^\top & 0 & 0 \\ * & -\tau P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & PH_d^\top & PG \\ * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -M & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -R & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\epsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\epsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\epsilon_3 I & 0 \\ * & * & * & * & * & * & * & * & * & * & -G \end{bmatrix} < 0, \quad (3.35a)$$

$$E_\alpha(\tau T^\alpha) \sum_{j=0}^{[T/\tau]+1} e^{\lambda \tau j} (E_\alpha(\tau T^\alpha) - 1)^j \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \leq \frac{c_2}{c_1}, \quad (3.35b)$$

then, under the state-feedback control law $u(t) = YP^{-1}x(t)$, system (3.30) is robustly finite-time stable with respect to (c_1, c_2, T) . Moreover, the guaranteed cost is given by:

$$J^* = \|\varphi\|^2 \frac{1}{\lambda_{\min}(P)} + \frac{2\gamma\tau}{\lambda_{\min}(P)} \|\varphi\|^2 \cdot \frac{T^\alpha}{\Gamma(\alpha + 1)},$$

where

$$\Theta_{11} = AP + PA^\top + BY + Y^\top B^\top + \epsilon_1 E_a E_a^\top + \epsilon_3 E_d E_d^\top + \epsilon_2 E_b E_b^\top - \tau P.$$

3.3 Conclusion of Chapter 3

This chapter investigated finite-time guaranteed cost control problems for two classes of fractional-order systems, including Caputo fractional-order convex combination systems with time-varying delays and uncertain nonlinear tempered Caputo fractional-order systems with constant delays.

- For Caputo fractional-order convex combination systems with time-varying delays, sufficient conditions for finite-time guaranteed cost control were established in terms of linear matrix inequalities based on the Lyapunov functional approach.
- For uncertain nonlinear tempered Caputo fractional-order systems with constant delays, a new lemma related to the tempered Caputo derivative was proposed. Based on this result, sufficient conditions ensuring finite-time guaranteed cost control were derived in terms of linear matrix inequalities by employing the Lyapunov functional method.
- Numerical examples and simulations were provided to demonstrate the feasibility of the proposed conditions. The simulation results verified the effectiveness and validity of the obtained theoretical criteria.

CONCLUSION

This dissertation investigated finite-time stability for several classes of fractional-order differential systems and related control problems.

Main Contributions of the Dissertation

- Some sufficient conditions were established for finite-time dissipativity of uncertain nonlinear tempered Caputo fractional-order neural networks with constant delays (Theorem 2.1.5), as well as for finite-time extended dissipativity of uncertain nonlinear Caputo fractional-order neural networks with disturbances (Theorem 2.2.6).
- Some sufficient conditions were derived for finite-time guaranteed cost control of Caputo fractional-order convex combination systems with time-varying delays (Theorem 3.1.4), as well as uncertain nonlinear tempered Caputo fractional-order systems with constant delays (Theorem 3.2.5).

Future Research Directions

Based on the results obtained in this dissertation, further studies may be developed on finite-time stability, dissipativity, passivity, guaranteed cost control, and H_∞ control for other classes of fractional-order dynamical systems, such as two-dimensional systems, large-scale systems, and switched systems. These problems can be investigated by combining the Lyapunov functional approach with tools from fractional calculus and matrix inequality techniques.

List of published works related to the thesis

- [CT1] T.N. Tuan, N.T. Thanh, M.V. Thuan (2023), “New results on robust finite-time extended dissipativity for uncertain fractional-order neural networks”, *Neural Processing Letters*, **55**, pp 9635–9650 (Q3/WoS, Q2/Scimago).
- [CT2] N.T. Thanh, M.V. Thuan, T.N. Tuan (2023), “New results on finite-time guaranteed cost control of uncertain polytopic fractional-order systems with time-varying delays”, *Optimal Control Applications and Methods*, **44**(5) pp 2504-2516 (Q2/WoS, Q2/Scimago).
- [CT3] T.N. Tuan, M.V. Thuan, N.T. Thanh (2025), “Finite-time guaranteed cost control for tempered nonlinear fractional-order systems with delays: A novel stability approach”, *Computational and Applied Mathematics*, **44**, Article 335(Q1/WoS, Q2/Scimago).
- [CT4] T.N. Tuan, N.T. Thanh, M.V. Thuan, N.T.T. Huyen (2026), “Finite-time dissipativity analysis of uncertain tempered fractional-order neural networks with delays via LMI approach”, *Journal of Applied Mathematics and Computing*, **72**, Article 59 (Q1/WoS, Q2/Scimago).