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QUALITATIVE PROPERTIES OF SOME SINGULAR SYSTEMS WITH INTEGER-ORDER AND FRACTIONAL-ORDER DERIVATIVE

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Introduction

Fractional calculus, with its two fundamental operations—differentiation and integration of arbitrary real or complex order—is regarded as an extension and generalization of concepts in classical calculus. Various types of fractional derivatives have been developed depending on how the nth-order derivative is generalized to the case where n is non-integer. Notable examples include the Riemann–Liouville derivative, the Caputo derivative, the Hadamard derivative, the Grünwald–Letnikov derivative, and the Marchaud derivative, ...

When the modeling process involves both differential and algebraic equations, the resulting system is a singular differential equation system. Stability is one of the most important qualitative properties, always emphasized when studying and analyzing dynamical systems. In the stability analysis of singular systems, two primary research directions are often considered: Lyapunov stability and finite-time stability. Lyapunov stability concerns the behavior of the state trajectory over an infinite time horizon, with the main objective of establishing conditions under which the system asymptotically returns to an equilibrium state. In contrast, finite-time stability emphasizes the ability of the system to maintain its stability within a specified finite time interval. The study of qualitative properties of singular differential equations presents greater challenges compared to regular differential equations. The existence of solutions for singular systems is not always guaranteed, even in the linear case.

In control theory, positive systems, where both state and output remain nonnegative given nonnegative initial conditions, are frequently encountered. As is well known, time delays commonly arise in various scientific and engineering applications, often degrading performance or even destabilizing the system. To the best of our knowledge, the problem of exponential stability for discrete-time impulsive singular positive systems with time delays has not yet been thoroughly investigated.

In practical applications, monitoring and regulating the behavior of the state vector described by differential equations over a finite time interval is crucial. In

1961, P. Dorato published some initial results on finite-time stability, and since then, the topic has received widespread attention in the control theory community. A differential system subjected to disturbances is said to be finite-time bounded if, given initial conditions that do not exceed a certain threshold and disturbances within a specified range, the system's state vector remains within a bounded region for a finite time interval.

The performance of a dynamical system is often evaluated through the relationship between input and output parameters, providing insight into the system's ability to optimize operational requirements. Several other important qualitative problems for singular integer-order and fractional-order differential systems have been proposed for further study, such as the dissipation problem, the guaranteed cost control problem, and mixed H_∞ and passive control problems, ...

First introduced in 1972 by J.C. Willems, dissipation theory provides an effective framework for stability analysis and control system design. One key property of dissipative systems is that the total energy contained in the system always decreases over time. With the Lyapunov theory as a fundamental tool, supported by linear matrix inequality (LMI) techniques, numerous significant results on the dissipation problem have been published for different classes of dynamical systems.

In control theory, one of the primary goals in system design is to optimize certain performance criteria, often referred to as "costs." The study of control functions that ensure not only system stability but also meet specific performance requirements is an essential aspect of control theory. The concept of guaranteed cost control (GCC) was introduced by S.S.L. Chang in 1969. In 1972, S.S.L. Chang and T.K.C. Keng formulated the optimal control problem along with a cost function for control systems. With the support of Lyapunov theory and LMI techniques, several significant results have been published on control cost optimization for various classes of dynamical systems.

Although extensive research has been conducted on singular differential systems with both integer-order and fractional-order derivatives, many essential qualitative problems for singular differential equations remain insufficiently explored. The problem of exponential stability analysis for discrete-time impulsive singular systems with time delays is complex and presents numerous unresolved technical challenges. Research on dissipation problems and the finite-time guaranteed cost control (FTGCC) problem has primarily focused on systems with integer-order derivatives, with some results for fractional differential systems, but very few studies have been published on singular fractional-order systems.

Based on the above analysis, we have chosen to investigate the qualitative properties of certain singular systems, focusing on key qualitative problems, including exponential stability and stabilization of discrete-time impulsive singular systems with constant delays, the finite-time dissipative control (FTDC) problem for fractional-order singular systems satisfying one-sided Lipschitz conditions with uncertainties, and the FTGCC problem for nonlinear fractional-order singular systems with disturbances.

In this thesis, we first propose and study the following discrete-time impulsive singular differential system with time delays

$$\begin{cases} Ex(t+1) = Ax(t) + A_d x(t-h),\ t \neq t_m - 1, \\ \mathtt{x}_1(t_m) = H\mathtt{x}_1(t_m - 1), t \in \mathbb{N}, \\ x(s) = \eta(s), s \in \{-h, -h+1, \cdots,\ 0\}, \end{cases}$$

where $x(\cdot) := (\mathtt{x}_1(\cdot),\ \mathtt{x}_2(\cdot))$, $\mathtt{x}_1(t) \in \mathbb{R}^r$ and $\mathtt{x}_2(t) \in \mathbb{R}^{n-r}$ is the state vector. The matrix $E \in \mathbb{R}^{n\times n}$ is singular and $\operatorname{rank}(E) = r < n$. $A,\ A_d, H$ are known constant matrices with appropriate dimensions. The delay constant $h \in \mathbb{N}^*$ and $\eta : [-h, 0] \to \mathbb{R}^n_+$ represent the initial condition. The impulse sequence $\{t_m\}_{m=1}^{\infty}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_m < \cdots, t_m \to \infty$ with $m \to \infty$. The results presented in this dissertation provide sufficient conditions for addressing the stability and exponential stabilization problems for the class of discrete-time impulsive singular positive systems with time delays.

The next class of systems under investigation is the singular Caputo fractional-order system satisfying a one-sided Lipschitz condition with uncertain parameters in the state vector.

$$\begin{cases} E\, {}^C_0D^{\alpha}_t x(t) = [A + \Delta A(t)]x(t) + [D + \Delta D(t)]\omega(t)F(t, x(t)) + Bu(t), t \geq 0, \\ z(t) = [C + \Delta C(t)]x(t) + W\omega(t), t \geq 0, \\ x(0) = x_0, \end{cases}$$

where $\alpha \in (0, 1)$ is the fractional order of the system, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $\omega(t) \in \mathbb{R}^q$ is the disturbance, $z(t) \in \mathbb{R}^p$ is the output vector, x_0 is the initial condition. $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times q}, W \in \mathbb{R}^{p\times q}$, and $E \in \mathbb{R}^{n\times n}$ are known constant matrices. The matrix E is singular with $\operatorname{rank}(E) = r \leq n$. Matrices $\Delta A(t), \Delta C(t), \Delta D(t)$ are time-varying and $\omega(.) \in L^2([0, +\infty), \mathbb{R}^q)$ satisfying the following condition

$$\exists d > 0 : \sup_{t\geq 0} \omega^T(t)\omega(t) \leq d,\ \forall t \in [0, T_f].$$

A key result of this dissertation is the design of a state feedback control function that ensures FTDC for the class of singular Caputo fractional-order systems satisfying a one-sided Lipschitz condition with uncertain parameters.

Next, we investigate the FTGCC problem for the class of singular Caputo fractional-order systems

$$\begin{cases} E\,{}_0^C D_t^{\alpha} x(t) = [A + \Delta A(t)]x(t) + [D + \Delta D(t)]\omega(t) + Bu(t), t \geq 0, \\ x(0) = x_0, \end{cases}$$

where the assumptions on the matrices and disturbance functions are similar to those in the second result. The final result of this dissertation is the design of a robust control function for the FTGCC problem of the singular Caputo fractional-order system with disturbances. In this dissertation, we aim to address the following problems:

- Investigate the exponential stability and exponential stabilization of the class of discrete-time impulsive singular positive systems with time delays and integer-order derivatives.

- Study finite-time boundedness for singular Caputo fractional-order systems satisfying a one-sided Lipschitz condition with uncertain parameters and design a state feedback control function for the FTDC control problem of the considered system.

- Develop a robust state feedback control function for the class of singular Caputo fractional-order systems with uncertain parameters that are finite-time bounded and satisfy control value constraints.

In addition to the introduction, conclusion and list of references, the thesis is presented in four chapters.

Chapter 1: Mathematical foundation.

Chapter 2: Exponential stability and stabilization of discrete-time impulsive positive singular system with time delays.

Chapter 3: Finite-time dissipative control design for one-sided Lipschitz nonlinear singular Caputo fractional order systems.

Chapter 4: The finite-time guaranteed cost control problem for a class of perturbed singular fractional differential systems.

Chapter 1

Mathematical foundation

This chapter presents fundamental mathematical knowledge that serves as the basis for studying the problems discussed in the subsequent chapters. The contents of Chapter 1 include: fundamental concepts of fractional calculus theory, solutions of singular differential systems, stability and exponential stabilization problems, finite-time boundedness problems, and related control problems such as the dissipativity problem, the GCC problem, and several supporting lemmas used to prove the results in this thesis.

1.1 Some fundamental knowledge of fractional integrals and derivatives.

1.2 The stability problem of discrete-time singular differential systems with delays

1.3 Singular fractional differential systems.

1.4 The finite-time stability problem

1.5 The dissipativity problem of dynamical systems.

1.6 The guaranteed cost control problem

1.7 Some supporting lemmas

Chapter 2

Exponential stability and stabilization of discrete-time impulsive positive singular system with time delays

This chapter focuses on the stability analysis of discrete-time impulsive singular positive systems with constant delays. We provide sufficient conditions to address the problem of exponential stability and design a control function for the exponential stabilization of discrete-time impulsive singular positive systems with time delays. The content of this chapter is based on the paper (CT1) in the List of published works related to the thesis.

2.1 Exponential stability of discrete-time impulsive positive singular system with time delayss

Consider the following impulsive singular systems

$$\begin{cases} Ex(t+1) & = Ax(t) + A_d x(t-h),\ t \neq t_m - 1, \\ \mathtt{x}_1(t_m) & = H\mathtt{x}_1(t_m - 1), \quad m \in \mathbb{Z}^+, \\ x(s) & = \eta(s), \quad s \in \{-h, -h+1, \cdots, 0\}, \end{cases} \tag{2.1}$$

where $x(\cdot) := (\mathtt{x}_1(\cdot),\ \mathtt{x}_2(\cdot))$, $\mathtt{x}_1(t) \in \mathbb{R}^r$ and $\mathtt{x}_2(t) \in \mathbb{R}^{n-r}$ is the state vector. The matrix $E \in \mathbb{R}^{n\times n}$ is singular and $\text{rank}(E) = r < n$. $A,\ A_d$ are known constant matrices with appropriate dimensions. In this paper, we suppose that the matrices E, A, A_d have the following expression:

$$E := \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, A := \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},\ A_d := \begin{pmatrix} A_{d_1} & A_{d_2} \\ A_{d_3} & A_{d_4} \end{pmatrix},$$

$A_1, A_{d_1} \in \mathbb{R}^{r\times r}, A_2, A_{d_2} \in \mathbb{R}^{r\times(n-r)}, A_3, A_{d_3} \in \mathbb{R}^{(n-r)\times r}, A_4,\ A_{d_4} \in \mathbb{R}^{(n-r)\times(n-r)}$, $H \in \mathbb{R}^{r\times r}$. The delay $h \in \mathbb{N}^*$. $\eta : [-h, 0] \to \mathbb{R}^n_+$ is a vector-valued initial function.

The impulse sequence $\{t_m\}_{m=1}^{\infty}$ satisfies $0 = t_0 < t_1 < \cdots < t_m < \cdots, t_m \to \infty$ for $m \to \infty$. Let us denote the state trajectory with the initial value (η_1, η_2) of the system (2.1) by $\mathtt{x}_1(t, \eta_1, \eta_2)$ and $\mathtt{x}_2(t, \eta_1, \eta_2)$.

Definition 2.1.1. System (2.1) is said to be an impulsive positive system if for all $\eta(t) \succeq 0$, then $\mathtt{x}(t, \eta(t)) \succeq 0$ for all $t \geq 0$.

Definition 2.1.2. The system (2.1) is said to be exponentially stable if $\exists\, \xi > 0$, $\alpha \in (0, 1), \lambda \in \mathbb{R}^n_+$ such that for any $\eta(\cdot) \succeq 0$,

$$\|x(t)\| \leq \xi \alpha^t \|\eta\|_h^{\dagger}, t \geq 0,$$

where $\|\eta\|_h^{\dagger} := \sup_{-h \leq s \leq 0} \|\eta(s)\|_{\infty}^{\lambda}$.

Lemma 2.1.3. *Assume that the matrix A_4 in system (2.1) satisfies the condition $\det(A_4) \neq 0$. Then, the solution of system (2.1) exists and is unique on $\mathbb{N}^*$.*

Lemma 2.1.4. *The system (2.1) is an impulsive positive system if A_4 is a Metzler and Hurwitz matrix and A_1, A_2, A_3, H, A_d are nonnegative matrices.*

We always assume that the matrices $A_E = A + I_n - E$, A_d, and H are nonnegative. A sufficient condition for examining the exponential stability of the system is given in the following theorem.

Theorem 2.1.5. *Suppose there exist numbers $\alpha \in (0, 1)$, $\delta \in (0, 1)$ and $\lambda \in \mathbb{R}^n_+$ satisfy the following constraints*

$$\left(-\alpha E + A + \alpha^{-h} A_d\right) \lambda \prec 0, \tag{2.2a}$$

$$R_\lambda^i = \frac{1}{\alpha} \sum_{j=1}^{r} h_{ij} \frac{\lambda_j}{\lambda_i} > 1, i \in J_{1,r}, \tag{2.2b}$$

$$\underline{\mathfrak{T}} \geq -\frac{1}{\delta} \log_\alpha R_\lambda, \tag{2.2c}$$

where $R_\lambda = \max_{1 \leq i \leq r} \left\{R_\lambda^i\right\}$. then, under the minimum dwell-time $\underline{\mathfrak{T}}$ (i.e., the impulse time sequence fulfills $\inf_m \{t_m - t_{m-1}\} \geq \underline{\mathfrak{T}}, m \in \mathbb{Z}^+$), system (2.1) is exponentially stable. Moreover, we have

$$\|x(t)\| \leq \|\eta\|_h^{\dagger} \|\lambda\|\, \alpha^{(1-\delta)t},\ t \geq 0.$$

2.2 Exponential stabilization of discrete-time impulsive singular positive systems with delays

Here, we are examining the following control system

$$\begin{cases} Ex(t+1) &= Ax(t) + A_d x(t-h) + Bu(t),\ t \neq t_m - 1, \\ \mathtt{x}_1(t_m) &= H\mathtt{x}_1(t_m - 1), t \in \mathbb{N}, \\ x(s) &= \eta(s), s \in \{-h, -h+1, \cdots,\ 0\}, \end{cases} \tag{2.3}$$

where $x(\cdot) := (\mathtt{x}_1(\cdot),\ \mathtt{x}_2(\cdot))$, $\mathtt{x}_1(t) \in \mathbb{R}^r$ and $\mathtt{x}_2(t) \in \mathbb{R}^{n-r}$ is the state vector, $u(t) \in \mathbb{R}^p$ is the control input, $B \in \mathbb{R}^{n \times p}$. In the case of the discrete-time impulsive positive singular system (2.3), when applying a state-feedback controller as given by

$$\begin{cases} Ex(t+1) &= (A + BK)x(t) + A_d x(t-h),\ t \neq t_m - 1, \\ \mathtt{x}_1(t_m) &= H\mathtt{x}_1(t_m - 1), t \in \mathbb{N}, \\ x(s) &= \eta(s), s \in \{-h, -h+1, \cdots,\ 0\}. \end{cases} \tag{2.4}$$

Theorem 2.2.1. *Suppose there exist numbers* $\alpha \in (0,1), \delta \in (0,1),\ \lambda \in \mathbb{R}^n_+$, $k_j \in \mathbb{R}^p, j = 1, 2, \ldots, n$ *satisfy the following constraints*

$$(A + I_n - E)_{(i,j)} \lambda_j + (B)_i^T k_j \geq 0, \quad i, j = 1, 2, \ldots, n, \tag{2.5a}$$

$$\left(-\alpha E + A + \alpha^{-h} A_d\right) \lambda + B \sum_{i=1}^{n} k_i \prec 0, \tag{2.5b}$$

$$R_\lambda^i = \frac{1}{\alpha} \sum_{j=1}^{r} h_{ij} \frac{\lambda_j}{\lambda_i} > 1, i \in J_{1,r}, \tag{2.5c}$$

$$\underline{\mathfrak{T}} \geq -\frac{1}{\delta} \log_\alpha R_\lambda, \tag{2.5d}$$

where $R_\lambda = \max_{1 \leq i \leq r} \left\{R_\lambda^i\right\}$. *then, under the minimum dwell-time* $\underline{\mathfrak{T}}$ *(i.e., the impulse time sequence fulfills* $\inf_m \{t_m - t_{m-1}\} \geq \underline{\mathfrak{T}}, m \in \mathbb{Z}^+$*), closed-loop system* (2.4) *is exponentially stable. Moreover, the feedback controller gain is obtained as*

$$K = \begin{bmatrix} \frac{k_1}{\lambda_1} & \frac{k_2}{\lambda_2} & \cdots & \frac{k_n}{\lambda_n} \end{bmatrix}.$$

2.3 Numerical examples

In this section, we present three examples to illustrate the effectiveness of the obtained results. Example 2.3.1 applies the result of Theorem 2.1.5 to examine the exponential stability of singular positive systems. Example 2.3.2 applies the result of Corollary 2.1.9 to analyze the exponential stability of nonsingular positive systems, with a comparison to the results of G. Zhang et al. (2012). Finally, Example 2.3.3 applies the result of Theorem 2.2.1 to address the problem of exponential stabilization. In this summary section, we briefly present Example 2.3.3.

Example 2.3.3 Consider the following control system (2.3), in which:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 3/25 & 1/10 & -1/10 \\ 1/10 & -1/10 & 1/10 \\ 1/10 & 0 & -4 \end{bmatrix}, \quad A_d = \begin{bmatrix} 1/5 & 1/10 & 0 \\ 1/10 & 1/10 & 0 \\ 3/20 & 1/10 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 11/100 & 1/5 \\ 1/5 & 1/10 \\ 0 & -1/10 \end{bmatrix}, H = \begin{bmatrix} 1/5 & 1 \\ 1 & 1/10 \end{bmatrix}.$$

For $\alpha = 0.9$ and $\delta = 0.8$, solving the conditions (2.5a)-(2.5c), we find the following values $k_1 = (\frac{1}{100}, \frac{1}{50})$, $k_2 = (\frac{1}{10}, 0)$, $k_3 = (\frac{1}{100}, \frac{1}{100})$, $\lambda = (\frac{3}{20}, \frac{3}{20}, \frac{3}{200})$, $R_\lambda^1 = 1.2 > 1$, $R_\lambda^2 = 1.1 > 1$, and $R_\lambda = 1.2 > 1$. Through some simple calculations, we obtain that $A + BK + I_3 - E = \begin{bmatrix} 131/5671 & 13/500 & 1/625 \\ 19/1000 & 1/200 & 9/2000 \\ 13/1000 & 0 & 939/1000 \end{bmatrix}$, and $A_d = \begin{bmatrix} 1/5 & 1/10 & 0 \\ 1/10 & 1/10 & 0 \\ 3/20 & 1/10 & 0 \end{bmatrix}$ are non-negative matrices and $K = \begin{bmatrix} 1/15 & 2/3 & 2/3 \\ 2/15 & 0 & 2/3 \end{bmatrix}$.

Hence, if we have time intervals $t_m - t_{m-1} \geq \mathfrak{T} \geq -\frac{1}{\delta}\log_\alpha R_\lambda \approx 2.1631, m \in \mathbb{Z}^+$, then, based on Theorem 2.2.1, the closed system is positive and exponentially stable.

The simulations start with an initial condition of $\nu(s) = (2\,4\,6)^T$, $s \in \{-2, -1, 0\}$. Figures 2.6 and 2.7 of the dissertation depict simulations of state trajectories $x_1(t)$, $x_2(t)$, and $x_3(t)$ of closed-loop system. These trajectories correspond to two cases: one where the system in Example 2.3.3 operates with impulses $t_m - t_{m-1} = 1$, and another with impulses $t_m - t_{m-1} = 3$, respectively.

Chapter 3

Finite-time dissipative control design for one-sided Lipschitz nonlinear singular Caputo fractional order systems

In this chapter, we design a control function for the finite-time dissipativity problem of a class of singular fractional-order systems satisfying the one-sided Lipschitz condition. The content of this chapter is developed based on the results presented in paper [CT2] in the List of published works related to the thesis.

3.1 Problem statement and some definitions

Consider the following one-sided Lipschitz singular systems described by

$$\begin{cases} E\,{}_0^C D_t^{\alpha} x(t) &= [A+\Delta A(t)]x(t) + [D+\Delta D(t)]\omega(t) \\ & \quad +F(t,x(t)) + Bu(t), t \geq 0, \\ z(t) &= [C+\Delta C(t)]x(t) + W\omega(t), t \geq 0, \\ x(0) &= x_0, \end{cases} \tag{3.1}$$

where $\alpha \in (0,1)$ is the fractional order of the system, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $\omega(t) \in \mathbb{R}^q$ is the disturbance, $z(t) \in \mathbb{R}^p$ is the output vector, x_0 is the initial condition. $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times q}, W \in \mathbb{R}^{p\times q}$, and $E \in \mathbb{R}^{n\times n}$ are known constant matrices. The matrix $E \in \mathbb{R}^{n\times n}$ is singular with $\text{rank}(E) = r \leq n$. Matrices $\Delta A(t), \Delta C(t), \Delta D(t)$ are time-varying, which represent time-varying parameter uncertainties as follows:

$$\Delta A(t) = M_a \mathcal{F}_a(t) N_a,\ \Delta C(t) = M_c \mathcal{F}_c(t) N_c,\ \Delta D(t) = M_d \mathcal{F}_d(t) N_d, \tag{3.2}$$

where $M_a, N_a, M_c, N_c, M_d, N_d$ are known constant real matrices and matrices $\mathcal{F}_a(t)$, $\mathcal{F}_c(t)$, $\mathcal{F}_d(t)$ are unknown real matrices satisfying

$$\mathcal{F}_a^T(t)\mathcal{F}_a(t) \leqslant I, \mathcal{F}_c^T(t)\mathcal{F}_c(t) \leqslant I,\ \mathcal{F}_d^T(t)\mathcal{F}_d(t) \leqslant I,\ \forall t \geqslant 0.$$

Assumption 3.1.1. $\omega(.) \in L^2([0,+\infty),\mathbb{R}^q)$ *satisfying the following sondition*

$$\exists d > 0 : \sup_{t\geq 0} \omega^T(t)\omega(t) \leq d.$$

$F(t,x(t)) = \Delta f(t,x(t)) + f(t,x(t))$, where $\Delta f(t,x(t))$ is an unknown nonlinear component. $f(t,x(t))$ satisfying $f(t,0) = 0$ and the following assumption.

Assumption 3.1.2. *The function $f(t,x(t))$ satisfies the one-sided Lipschitz condition with respect to $x(t)$ in the region $\mathfrak{D}$, i.e there exists constant $\rho \in \mathbb{R}$ such that*

$$\langle f(t,x_1(t)) - f(t,x_2(t)), x_1(t) - x_2(t)\rangle \leq \rho\|x_1(t) - x_2(t)\|^2,$$

$\forall x_1(t), x_2(t) \in \mathfrak{D}$. *The scalar ρ is called the one-sided Lipschitz constant.*

Assumption 3.1.3. *The function $f(t,x(t))$ satisfies the quadratic inner-bounded condition with respect to $x(t)$, i.e there exist constants $\mu,\nu \in \mathbb{R}$ such that*

$$\begin{aligned}\|f(t,x_1(t)) - f(t,x_2(t))\|^2 \leq &\mu\langle x_1(t) - x_2(t), f(t,x_1(t)) - f(t,x_2(t))\rangle \\ &+ \nu\|x_1(t) - x_2(t)\|^2,\end{aligned}$$

for all $x_1(t), x_2(t) \in \mathfrak{D}$.

Assumption 3.1.4. $\Delta f(t,x(t))$ *satisfying the following condition*

$$\|\Delta f(t,x(t))\| \leq h\|x(t)\|, \forall x(t) \in \mathfrak{D},$$

where h is a given positive scalar.

In the absence of control input vector, the system (3.1) becomes

$$\begin{cases} E\,{}^C_0D^\alpha_t x(t) &= [A + \Delta A(t)]x(t) + [D + \Delta D(t)]\omega(t) + F(t,x(t)), t \geq 0, \\ z(t) &= [C + \Delta C(t)]x(t) + W\omega(t), t \geq 0, \\ x(0) &= x_0. \end{cases} \tag{3.3}$$

In the absence of output vector, the system (3.3) becomes

$$\begin{cases} E\,{}^C_0D^\alpha_t x(t) &= [A + \Delta A(t)]x(t) + [D + \Delta D(t)]\omega(t) + F(t,x(t)), t \geq 0, \\ x(0) &= x_0. \end{cases} \tag{3.4}$$

Definition 3.1.5. The systems (3.4) is
i) regular if exits $s \in \mathbb{C}$ such that the polynomial $\det(sE - (A + \Delta A(t)))$ is not identically zero. Then the matrix pair (E, A) is called regular.
ii) impulse-free if $\deg(\det(sE - (A + \Delta A(t)))) = \text{rank(E)}$, for some $s \in \mathbb{C}$.

Definition 3.1.6. For given positive scalars c_1, c_2, T_f with $c_1 < c_2$ and a symmetric positive definite matrix R. System (3.4) is said to be finite-time boundedness with respect to (w.r.t.) (c_1, c_2, T_f, R, d) if it is regular, impulse-free and the following relation holds

$$x_0^T E^T R E x_0 \leq c_1 \Rightarrow x^T(t) E^T R E x(t) < c_2, \ \forall t \in [0, T_f]. \tag{3.5}$$

Definition 3.1.8 For given positive scalars c_1, c_2, T_f with $c_1 < c_2$ and a symmetric positive definite matrix R. System (3.3) said to be roburst finite-time (Z, U, S)-dissipative w.r.t. (c_1, c_2, T_f, R, d) if two the following conditions are met:
(i) The system (3.4) is finite-time boundedness w.r.t. (c_1, c_2, T_f, R, d).
(ii) Consider the system (3.3). Under the zero-initial condition, $\exists \gamma > 0$ such that the following inequality holds:

$$\int_0^{t_f} (2z^T(t)U\omega(t) + z^T(t)Zz(t) + \omega^T(t)S\omega(t))dt \geq \gamma \int_0^{t_f} \omega^T(t)\omega(t)dt,$$

for $\forall t_f \in [0, T_f]$ and any non zero disturbance vector $\omega(t)$ satisfying Assumption 3.1.1, where Z, S are given real symmetry matrices with $Z \leqslant 0$, and U is a given real matrix.

The closed-loop system with $u(t) = Kx(t)$ is

$$\begin{cases} E\,{}_0^C D_t^\alpha x(t) &= [A + BK + \Delta A(t)]x(t) + [D + \Delta D(t)]\omega(t) \\ &\qquad +F(t, x(t)), t \geq 0, \\ z(t) &= [C + \Delta C(t)]x(t) + W\omega(t), t \geq 0, \\ x(0) &= x_0. \end{cases} \tag{3.6}$$

In the absence of output vector, the closed-loop system (3.6) reduces

$$\begin{cases} E\,{}_0^C D_t^\alpha x(t) &= [A + BK + \Delta A(t)]x(t) + [D + \Delta D(t)]\omega(t) \\ &\qquad +F(t, x(t)), t \geq 0, \\ x(0) &= x_0. \end{cases} \tag{3.7}$$

3.2 Finite-time boundedness for one-sided Lipschitz nonlinear singular Caputo fractional order systems

Theorem 3.2.1. *Suppose that the assumptions 3.1.1-3.1.4 are satisfied. For given positive scalars* c_1, c_2, T_f *with* $c_1 < c_2$ *and a symmetric positive definite matrix* R.

The systems (3.7) *is finite-time boundedness w.r.t.* (c_1, c_2, T_f, R, d)*, if there exist scalars* $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon > 0, \delta > 0, \theta > 0, \sigma > 0, \beta > 0$*, a symmetric positive definite matrix* $\Sigma \in \mathbb{R}^{n\times n}$*, a non-singular matrix* $Q \in \mathbb{R}^{n\times n}$*, a non-singular matrix* $F \in \mathbb{R}^{m\times m}$*, and a matrix* $L \in \mathbb{R}^{m\times n}$ *such that the following conditions hold*

$$E^TQ = Q^TE \geq 0, \tag{3.8a}$$

$$E^TQ = E^TR^{\frac{1}{2}}\Sigma R^{\frac{1}{2}}E, \tag{3.8b}$$

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & Q^TD & \Xi_{14} & Q^TM_a & Q^TM_d & Q^T \\ * & -2\epsilon_2 I & 0 & 0 & 0 & 0 & 0 \\ * & * & \sigma N_d^TN_d - \beta I & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & 0 \\ * & * & * & * & -\theta I & 0 & 0 \\ * & * & * & * & * & -\sigma I & 0 \\ * & * & * & * & * & 0 & -\delta I \end{bmatrix} < 0, \tag{3.8c}$$

$$\lambda_1 c_1 + \frac{\beta d}{\Gamma(\alpha+1)}T_f^{\alpha} < \lambda_2 c_2, \tag{3.8d}$$

where

$$\begin{aligned}
\Xi_{11} &= A^TQ + Q^TA + L^TB^T + BL + \delta h^2 I + \theta N_a^TN_a + 2(\rho\epsilon_1 + \epsilon_2\nu)I,\\
\Xi_{12} &= Q^T + (\epsilon_2\mu - \epsilon_1)I,\\
\Xi_{14} &= \epsilon(Q^TB - BF) + L^T,\\
\Xi_{44} &= -\epsilon F - \epsilon F^T,\\
\lambda_1 &= \lambda_{\max}(\Sigma), \lambda_2 = \lambda_{\min}(\Sigma).
\end{aligned}$$

Moreover, the state feedback controller is given by $u(t) = F^{-1}Lx(t)$*.*

3.3 Finite-time dissipative control design for one-sided Lipschitz nonlinear singular Caputo fractional order systems

Theorem 3.3.1. *Suppose that the assumptions 3.1.1-3.1.4 are satisfied. For given positive scalars* c_1, c_2, T_f *with* $c_1 < c_2$ *and a symmetric positive definite matrix* R*.The closed-loop system* (3.6) *is roburst finite time* (Z, U, S)*-dissipative w.r.t.* (c_1, c_2, T_f, R, d) *if there exist positive scalars* $\epsilon_1, \epsilon_2, \epsilon, \beta, \delta, \theta, \sigma, \kappa, \gamma$*, a symmetric positive definite matrix* $\Sigma \in \mathbb{R}^{n\times n}$*, a non-singular matrix* $Q \in \mathbb{R}^{n\times n}$*, a non-singular matrix* $F \in \mathbb{R}^{m\times m}$*, and a matrix* $L \in \mathbb{R}^{m\times n}$ *such that* (3.8a)*,* (3.8b) *and the* *following conditions hold*

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & Q^T M_a & Q^T M_d & Q^T & C^T G^T & 0 \\ * & -2\epsilon_2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Xi_{33} & 0 & 0 & 0 & 0 & 0 & \Xi_{39} \\ * & * & * & \Xi_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\theta I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\sigma I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\delta I & 0 & 0 \\ * & * & * & * & * & * & * & -I & GM_c \\ * & * & * & * & * & * & * & * & -\kappa I \end{bmatrix} < 0, \tag{3.9a}$$

$$S - (\gamma + \beta) I > 0, \tag{3.9b}$$

$$\lambda_1 c_1 + \frac{\beta d}{\Gamma(\alpha+1)} T_f^{\alpha} < \lambda_2 c_2, \tag{3.9c}$$

where

$$\begin{aligned}
&\lambda_1 = \lambda_{\max}(\Sigma), \lambda_2 = \lambda_{\min}(\Sigma), Z = -G^T G, \\
&\Xi_{11} = A^T Q + Q^T A + L^T B^T + BL + \delta h^2 I + \theta N_a^T N_a + \kappa N_c^T N_c + 2(\rho\epsilon_1 + \epsilon_2 \nu) I, \\
&\Xi_{12} = Q^T + (\epsilon_2 \mu - \epsilon_1) I, \\
&\Xi_{14} = \epsilon(Q^T B - BF) + L^T, \\
&\Xi_{13} = Q^T D - C^T Z W - C^T U, \\
&\Xi_{33} = \sigma N_d^T N_d - W^T Z W - W^T U - U^T W - \beta I, \\
&\Xi_{39} = -(W^T Z^T + U^T) M_c, \\
&\Xi_{44} = -\epsilon F - \epsilon F^T.
\end{aligned}$$

Moreover, the state feedback controller is given by $u(t) = F^{-1} L x(t)$.

Theorem 3.3.2. *Suppose that the assumptions 3.1.1-3.1.4 are satisfied. For given positive scalars* c_1, c_2, T_f *with* $c_1 < c_2$ *and a symmetric positive definite matrix* R. *The closed-loop system* (3.6) *is roburst finite time* (Z, U, S)*-dissipative w.r.t.* (c_1, c_2, T_f, R, d) *if there exist scalars* $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon > 0, \beta > 0, \delta > 0, \theta > 0, \sigma > 0, \gamma > 0, \kappa > 0, \zeta > 0$, *a symmetric positive definite matrix* $\mathcal{R} \in \mathbb{R}^{n\times n}$, *a matrix* $\mathcal{S} \in \mathbb{R}^{(n-r)\times n}$, *a non-singular matrix* $F \in \mathbb{R}^{m\times m}$, *and a matrix* $L \in \mathbb{R}^{m\times n}$ *such that the following conditions*

$$R < \mathcal{R} < \zeta R, \tag{3.10a}$$

$$\zeta c_1 + \frac{\beta d}{\Gamma(\alpha+1)} T_f^{\alpha} < c_2, \tag{3.10b}$$

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} & \Gamma_{17} & C^T G^T & 0 \\ * & -2\epsilon_2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Gamma_{33} & 0 & 0 & 0 & 0 & 0 & \Gamma_{39} \\ * & * & * & \Gamma_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\theta I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\sigma I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\delta I & 0 & 0 \\ * & * & * & * & * & * & * & -I & GM_c \\ * & * & * & * & * & * & * & * & -\kappa I \end{bmatrix} < 0, \tag{3.10c}$$

where

$$\begin{aligned}
\Gamma_{11} &= A^T(\mathcal{R}E + E_\perp^T \mathcal{S}) + (E^T\mathcal{R} + \mathcal{S}^T E_\perp)A + L^T B^T + BL + \delta h^2 I \\
&\quad + \theta N_a^T N_a + \kappa N_c^T N_c + 2(\rho\epsilon_1 + \epsilon_2\nu)I, \\
\Gamma_{12} &= (E^T\mathcal{R} + \mathcal{S}^T E_\perp) + (\epsilon_2\mu - \epsilon_1)I, \\
\Gamma_{13} &= (E^T\mathcal{R} + \mathcal{S}^T E_\perp)D - C^T ZW - C^T U, \\
\Gamma_{14} &= \epsilon(E^T\mathcal{R} + \mathcal{S}^T E_\perp)B - \epsilon BF + L^T, \\
\Gamma_{15} &= (E^T\mathcal{R} + \mathcal{S}^T E_\perp)M_a, \\
\Gamma_{16} &= (E^T\mathcal{R} + \mathcal{S}^T E_\perp)M_d, \\
\Gamma_{17} &= (E^T\mathcal{R} + \mathcal{S}^T E_\perp), \\
\Gamma_{33} &= \sigma N_d^T N_d - W^T ZW - W^T U - U^T W - \beta I, \\
\Gamma_{39} &= -(W^T Z^T + U^T)M_c, \\
\Gamma_{44} &= -\epsilon F - \epsilon F^T,
\end{aligned}$$

$Z = -G^T G$, *and* $E_\perp \in \mathbb{R}^{(n-r)\times n}$ *is the orthogonal complement of* E *such that* $E_\perp E = 0$, *and* $rank(E_\perp) = n - r$. *Moreover, the state feedback controller is given by* $u(t) = F^{-1}Lx(t)$.

3.4 Numerical examples

In this section, we present two examples to illustrate the effectiveness of the obtained results. The Example 3.3.1 applies the result of theorem 3.3.2 to analyze the dissipativity problem for a singular fractional-order system with uncertain components. example 3.3.2 applies the result of corollary 3.3.5 to solve the FTDC problem in a special case of the singular fractional-order system. in this summary, we briefly present Example 3.3.1.

Example 3.4.1. Consider the following control system

$$\begin{cases} E\,{}_0^C D_t^{0.8} x(t) & = [A + M_a \mathcal{F}_a(t) N_a] x(t) + [D + M_d \mathcal{F}_d(t) N_d] \omega(t) \\ & \quad + f(t, x(t)) + \Delta f(t, x(t)) + Bu(t), t \geq 0, \\ z(t) & = [C + M_c \mathcal{F}_c(t) N_c] x(t) + W\omega(t), t \geq 0, \\ x(0) & = x_0, \end{cases} \tag{3.11}$$

where $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2, u(t) \in \mathbb{R}^2, \omega(t) \in \mathbb{R}$,

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}, M_a = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, N_a = \begin{bmatrix} 0.1 & 0.5 \end{bmatrix}, \mathcal{F}_a(t) = \sin t,$$
$$D = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, M_d = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, N_d = \begin{bmatrix} 1 \end{bmatrix}, \mathcal{F}_d(t) = \cos t,$$
$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, M_c = \begin{bmatrix} -0.3 & 0.2 \end{bmatrix}, N_c = \begin{bmatrix} 0.1 \end{bmatrix} \; W = \begin{bmatrix} 0.5 \end{bmatrix}, \mathcal{F}_c(t) = \sin t,$$
$$f(t, x(t)) = \begin{bmatrix} 0.1 x_1^2(t) \\ 0 \end{bmatrix}, \Delta f(t, x(t)) = \begin{bmatrix} x_1^2(t) \\ x_1(t) x_2(t) \end{bmatrix}.$$

The closed-loop system with a state feedback controller $u(t) = Kx(t)$ of system (3.11) is described by

$$\begin{cases} E\,{}_0^C D_t^{0.8} x(t) & = [A + M_a \mathcal{F}_a(t) N_a + BK] x(t) + [D + M_d \mathcal{F}_d(t) N_d] \omega(t) \\ & \quad + f(t, x(t)) + \Delta f(t, x(t)), t \geq 0, \\ z(t) & = [C + M_c \mathcal{F}_c(t) N_c] x(t) + W\omega(t), t \geq 0, \\ x(0) & = x_0. \end{cases} \tag{3.12}$$

It is not hard to check that $f(t, x(t))$ satisfies Assumptions 3.1.2 and 3.1.3 with the following constants $\rho = 1, \mu = 0, \nu = 1$ in the region $\mathfrak{D} = \{x \in \mathbb{R}^2 : \|x\| \leq 5\}$, and the nonlinear uncertainty $\Delta f(t, x(t))$ satisfies Assumption 3.1.4 with constant $h = 1$.

Given $c_1 = 1, c_2 = 1.7, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, d = 1, T_f = 10, U = \begin{bmatrix} 1 \end{bmatrix}, Z = \begin{bmatrix} -1 \end{bmatrix}$, $S = \begin{bmatrix} 1 \end{bmatrix}$. By using LMI Control Toolbox in MATLAB, the conditions in Theorem

3.3.2 are feasible with $\epsilon_1 = 0.7454$, $\epsilon_2 = 0.0882$, $\epsilon = 1, \beta = 0.5023$, $\delta = 4.0496$, $\theta = 4.0449, \sigma = 0.8417, \gamma = 0.2488, \kappa = 4.0865, \zeta = 5.9192$,

$$\mathcal{R} = \begin{bmatrix} 2.0942 & -0.0865 \\ -0.0865 & 2.8886 \end{bmatrix}, \mathcal{S} = \begin{bmatrix} 0.0912 & 1.7755 \end{bmatrix},$$

$$L = \begin{bmatrix} -3.6720 & 9.6760 \\ -3.5282 & 2.9514 \end{bmatrix}, F = \begin{bmatrix} 3.4008 & 1.0265 \\ -1.1638 & 1.4339 \end{bmatrix}.$$

According to Theorem 3.3.2, the closed-loop system (3.12) is roburst finite-time (Z, U, S)-dissipative w.r.t. $(1, 1.7, 10, R, 1)$ under the state feedback controller $u(t) = \begin{bmatrix} -0.2707 & 1.7863 \\ -2.6803 & 3.5081 \end{bmatrix} x(t), \forall t \in [0, 10]$.

For simulation results, we choose the initial condition $x_0 = (-1, 1)^T \in \mathbb{R}^2$, and the disturbance $\omega(t) = \sin t$. Figure 3.1 in the thesis presents the time history of $x^T(t)E^T REx(t)$ of the open-loop systems. It is obvious that the open-loop system is not SFTB since they also do not meet the condition of Definition 3.1.6. Figure 3.2. in the thesis shows the time history of $x^T(t)E^T REx(t)$ of the closed-loop system. It could be seen that the value of $x^T(t)E^T REx(t)$ is limited to less than $c_2 = 1.7$ in the given finite-time interval $t \in [0, 10]$. That is to say, the system (3.12) is FTB in terms of 3.1.6. To check the (Z, U, S)-dissipative performance in $[0, 10]$, we define a dissipativity performance function as

$$\gamma(t) = \frac{\int_0^{t_f} (2z^T(t)U\omega(t) + z^T(t)Zz(t) + \omega^T(t)S\omega(t))dt}{\int_0^{t_f} \omega^T(t)\omega(t)dt}, \forall t_f \in [0, 10]$$

Figure 3.3 in the thesis indicates the dissipativity performance function $\gamma(t)$ is always greater than $\gamma = 0.2488$. It is easy to see that the closed-loop system (3.12) is robustly finite-time (Z, U, S)-dissipative w.r.t. $(1, 1.7, 10, R, 1)$ with the scalar $\gamma = 0.2488$.

Chapter 4

The finite-time guaranteed cost control problem for a class of perturbed singular fractional differential systems

In this chapter, we design a robust control function for the FTGCC problem of perturbed Caputo fractional singular differential systems. The content of this chapter is developed based on the results presented in the paper [CT3] in the List of published works related to the thesis.

4.1 The finite-time guaranteed cost control problem

Consider the following singular Caputo fractional-order systems

$$\begin{cases} E\, {}^{C}_{0}D^{\alpha}_{t}x(t) &= [A+\Delta A(t)]x(t)+[D+\Delta D(t)]\omega(t)+Bu(t), t\geq 0,\\ x(0) &= x_0, \end{cases} \tag{4.1}$$

where $\alpha \in (0,1)$ is the fractional order of the system, $x(t)\in \mathbb{R}^n$ is the state vector, $u(t)\in \mathbb{R}^m$ is the control input vector, $\omega(t)\in\mathbb{R}^q$ is the disturbance, x_0 is the initial condition. $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times m}, D\in\mathbb{R}^{n\times q}$, and $E\in\mathbb{R}^{n\times n}$ are known constant matrices. The matrix $E\in\mathbb{R}^{n\times n}$ is singular with rank(E) $= r < n$. Matrices $\Delta A(t), \Delta D(t)$ are time-varying, which represent time-varying parameter uncertainties as follows:

$$\Delta A(t) = M_a\mathcal{F}_a(t)N_a,\ \Delta D(t) = M_d\mathcal{F}_d(t)N_d, \tag{4.2}$$

where M_a, N_a, M_d, N_d are known constant real matrices and matrices $\mathcal{F}_a(t), \mathcal{F}_d(t)$ are unknown real matrices satisfying

$$\mathcal{F}_a^T(t)\mathcal{F}_a(t)\leqslant I, \mathcal{F}_d^T(t)\mathcal{F}_d(t)\leqslant I,\ \forall t\geqslant 0.$$

Given a positive number $T_f > 0$. Associated with the singular system (4.1) is the following cost function

$$J = \int_0^{T_f} [x^T(t)S_1x(t) + u^T(t)S_2u(t)]dt, \tag{4.3}$$

where $S_1 > 0$ and $S_2 > 0$ are given known constant matrices with approriate dimensions.

Assumption 4.1.1. *The disturbance* $\omega(.) \in L^2([0, +\infty), \mathbb{R}^q)$ *satisfying the following condition*

$$\exists d > 0 : \sup_{t \geq 0} \omega^T(t)\omega(t) \leq d.$$

In the absence of control input vector, the system (4.1) becomes

$$\begin{cases} E\, {}_0^C D_t^\alpha x(t) & = [A + \Delta A(t)]x(t) + [D + \Delta D(t)]\omega(t), t \geq 0, \\ x(0) & = x_0. \end{cases} \tag{4.4}$$

Definition 4.1.2. The systems (4.4) is
i) regular if exits $s \in \mathbb{C}$ such that the polynomial $\det(sE - (A + \Delta A(t)))$ is not identically zero. Then the matrix pair (E, A) is called regular.
ii) impulse-free if $\deg(\det(sE - (A + \Delta A(t)))) = \text{rank(E)}$, for some $s \in \mathbb{C}$.

Definition 4.1.3. For given positive scalars c_1, c_2, T_f with $c_1 < c_2$ and a symmetric positive definite matrix R. System (4.4) is said to be finite-time boundedness with respect to (w.r.t.) (c_1, c_2, T_f, R, d) if it is regular, impulse-free and the following relation holds

$$x_0^T E^T R E x_0 \leq c_1 \Rightarrow x^T(t)E^T R E x(t) < c_2, \, \forall t \in [0, T_f]. \tag{4.5}$$

Consider the non-fragile state feedback controller in the following form

$$u(t) = (K + \Delta K(t))x(t), \tag{4.6}$$

in which $K \in \mathbb{R}^{m \times n}$ is the state-feedback gain matrix to be designed, and $\Delta K(t)$ is an a priori norm-bounded gain variation of the form $\Delta K(t) = M_k \mathcal{F}_k(t) N_k$, where M_k and N_k are known constant real matrices with appropriate dimensions and $\mathcal{F}_k(t)$ is unknown real matrix satisfying$\mathcal{F}_k^T(t)\mathcal{F}_k(t) \leqslant I, \; \forall t \geqslant 0$. Thus, the closed-loop system is obtained as follows

$$\begin{cases} E\, {}_0^C D_t^\alpha x(t) & = [A + \Delta A(t) + BK + B\Delta K(t)]x(t) \\ & \quad + [D + \Delta D(t)]\omega(t), t \geq 0, \\ x(0) & = x_0. \end{cases} \tag{4.7}$$

Definition 4.1.4. For given positive scalars c_1, c_2, T_f with $c_1 < c_2$ and a symmetric positive definite matrix R. Consider the system (4.1) and the cost function (4.3). If there exists a non-fragile state feedback control $\hat{u}(t)$ and a positive constant J^* such that the closed-loop system (4.7) is robustly SFTB w.r.t. (c_1, c_2, T_f, R, h), and the closed-loop value of the cost function satisfies $J \leq J^*$, then the value J^* is called guaranteed-cost, $\hat{u}(t)$ is called the guaranteed cost control function.

Theorem 4.1.5. *Consider system* (4.1) *with the cost function* (4.3). *For given positive scalars* c_1, c_2, T_f *with* $c_1 < c_2$ *and a symmetric positive definite matrix* R. *Suppose that assumption 4.1.1 is satisfied and exist scalars* $\delta > 0, \theta > 0, \beta > 0, \gamma > 0, \mu_1 > 0, \mu_2 > 0$, *a non-singular matrix* $P \in \mathbb{R}^{n\times n}$, *and a matrix* $Y \in \mathbb{R}^{n\times m}$ *such that the following conditions hold*

$$PE^T = EP^T \geq 0, \tag{4.8a}$$

$$\mu_1 PE^T < PE^T REP^T < \mu_2 PE^T, \tag{4.8b}$$

$$\begin{bmatrix} \Omega_{11} & PN_a^T & PN_k^T & D & M_d & PS_1 & YS_2 & YS_2M_k & PN_k^T \\ * & -\delta I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\theta I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\beta I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -S_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & -S_2 & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -\sigma I \end{bmatrix} < 0, \tag{4.8c}$$

$$\frac{c_1}{\mu_1} + \frac{(\beta + \gamma\lambda_{\max}(N_d^T N_d))d}{\Gamma(\alpha+1)} T_f^\alpha < \frac{c_2}{\mu_2}, \tag{4.8d}$$

where

$$\Omega_{11} = PA^T + YB^T + AP^T + BY^T + \delta M_a M_a^T + \theta B M_k M_k^T B^T,$$

$$\sigma = \frac{1}{1 + \lambda_{\max}(S_2)\lambda_{\max}(M_k^T M_k)}.$$

then (4.6) *is a non-fragile state feedback finite-time guaranteed cost controller for* (4.1). *Moreover, the state feedback matrix* K *is determined by* $K = Y^T P^{-T}$, *and The guaranteed cost value* J^* *can be expressed as*

$$J^* = (\beta + \gamma\lambda_{\max}(N_d^T N_d))dT_f + \frac{T_f^{1-\alpha}}{\Gamma(2-\alpha)}\lambda_{\max}(E^T P^{-T})\|x_0\|^2.$$

Theorem 4.1.6. *Consider system (4.1) with the cost function (4.3). For given positive scalars c_1, c_2, T_f with $c_1 < c_2$ and a symmetric positive definite matrix R. Suppose that assumption 4.1.1 is satisfied and there exist positive scalars δ, θ, β, γ, μ_1, μ_2, a symmetric positive definite matrix $\hat{\mathcal{X}} \in \mathbb{R}^{r\times r}$, a matrix $\hat{\mathcal{Z}} \in \mathbb{R}^{n\times(n-r)}$ and a matrix $Y \in \mathbb{R}^{n\times m}$ such that the following conditions hold*

$$\mu_1\Xi < \hat{\mathcal{X}} < \mu_2\Xi, \tag{4.9a}$$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & D & M_d & \Pi_{16} & YS_2 & YS_2M_k & \Pi_{19} \\ * & -\delta I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\theta I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\beta I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -S_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & -S_2 & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -\sigma I \end{bmatrix} < 0, \tag{4.9b}$$

$$\frac{c_1}{\mu_1} + \frac{(\beta + \gamma\lambda_{\max}(N_d^T N_d))d}{\Gamma(\alpha+1)} T_f^{\alpha} < \frac{c_2}{\mu_2}, \tag{4.9c}$$

where

$$\begin{aligned} \Pi_{11} &= (EV_1\hat{\mathcal{X}}V_1^T + \hat{\mathcal{Z}}V_2^T)A^T + A(V_1\hat{\mathcal{X}}V_1^T E^T + V_2\hat{\mathcal{Z}}^T) + YB^T + BY^T \\ &\quad + \delta M_a M_a^T + \theta B M_k M_k^T B^T, \\ \Pi_{12} &= (EV_1\hat{\mathcal{X}}V_1^T + \hat{\mathcal{Z}}V_2^T)N_a^T, \\ \Pi_{13} &= (EV_1\hat{\mathcal{X}}V_1^T + \hat{\mathcal{Z}}V_2^T)N_k^T, \\ \Pi_{16} &= (EV_1\hat{\mathcal{X}}V_1^T + \hat{\mathcal{Z}}V_2^T)S_1, \\ \Pi_{19} &= (EV_1\hat{\mathcal{X}}V_1^T + \hat{\mathcal{Z}}V_2^T)N_k^T, \\ \Xi &= (\Sigma_r U_1^T R U_1 \Sigma_r)^{-1}, \\ \sigma &= \frac{1}{1 + \lambda_{\max}(S_2)\lambda_{\max}(M_k^T M_k)}. \end{aligned}$$

then (4.6) is a non-fragile state feedback finite-time guaranteed cost controller for (4.1). The state feedback matrix K is determined by $K = Y^T(EV_1\hat{\mathcal{X}}V_1^T + \hat{\mathcal{Z}}V_2^T)^{-T}$. The guaranteed cost value J^* is determined by

$$J^* = (\beta + \gamma\lambda_{\max}(N_d^T N_d))dT_f + \frac{T_f^{1-\alpha}}{\Gamma(2-\alpha)}\lambda_{\max}(E^T(EV_1\hat{\mathcal{X}}V_1^T + \hat{\mathcal{Z}}V_2^T)^{-T})\|x_0\|^2.$$

4.2 Numerical examples

In this section, we present two examples to illustrate the effectiveness of the obtained results. Example 4.2.1 applies the result of Theorem 4.1.6 to analyze the FTGCC problem for a singular fractional-order system with uncertain components. Example 4.2.2 applies the result of Corollary 4.1.9 to solve the FTDC problem in a special case of a singular fractional-order system. In this summary, we briefly present Example 4.2.1.

Example 4.2.1. Consider the following singular fractional-order systems

$$\begin{cases} E\,{}_0^C D_t^{0.8} x(t) &= [A + M_a \mathcal{F}_a(t) N_a]x(t) + [D + M_d \mathcal{F}_d(t) N_d]\omega(t) \\ & \quad +Bu(t), t \geq 0, \\ x(0) &= x_0, \end{cases} \tag{4.10}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$ is the state vector, $u(t) \in \mathbb{R}^2$ is the control, $\omega(t) \in \mathbb{R}$ is the disturbance, and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix},$$

$$M_a = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.5 \end{bmatrix}, M_d = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.7 \end{bmatrix},$$

$$N_a = \begin{bmatrix} 0.1 & 0.9 & -0.5 \end{bmatrix}, \mathcal{F}_a(t) = \sin t,\ N_d = \begin{bmatrix} 1 \end{bmatrix}, \mathcal{F}_d(t) = \cos t.$$

The closed-loop system described by the non-fragile state feedback controller with $u(t) = [K + M_k \mathcal{F}_k(t) N_k]x(t)$, $M_k = \begin{bmatrix} -0.2 \\ 0.3 \end{bmatrix}$, $N_k = \begin{bmatrix} 0.2 & -0.1 & 0.3 \end{bmatrix}$, and $\mathcal{F}_k(t) = \cos t$, can be represent as follows

$$\begin{cases} E\,{}_0^C D_t^{0.8} x(t) &= [A + BK + M_a \mathcal{F}_a(t) N_a + BM_k \mathcal{F}_k(t) N_k]x(t) \\ & \quad +[D + M_d \mathcal{F}_d(t) N_d]\omega(t), t \geq 0, \\ x(0) &= x_0. \end{cases} \tag{4.11}$$

The cost function associated with the system (4.10) is given in (4.3) with

$$S_1 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, S_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Given $c_1 = 1, c_2 = 9, T_f = 30$, and $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we consider the problem of finite-time guaranteed cost control of the system (4.10). Using the svd command in MATLAB, we compute the matrices U_1, U_2, V_1, V_2, and Σ_r in Theorem 4.1.6 as follows

$$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, U_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

With the help of the LMI Control Toolbox in MATLAB, we verified that the conditions (4.9a), (4.9b) and (4.9c) are satisfied with $\delta = 1.0159, \theta = 1.0339$, $\beta = 1.1988, \gamma = 1.2260, \mu_1 = 0.2220, \mu_2 = 1.5746$,

$$\hat{\mathcal{X}} = \begin{bmatrix} 0.7841 & -0.0202 \\ -0.0202 & 0.6787 \end{bmatrix}, \hat{\mathcal{Z}} = \begin{bmatrix} -0.0735 \\ 0.4615 \\ 0.7210 \end{bmatrix}, Y = \begin{bmatrix} -0.0001 & 0.1079 \\ 0.0927 & -0.8564 \\ 0.2857 & 0.5997 \end{bmatrix}.$$

According to Theorem 4.1.6, the closed-loop system (4.11) is finite-time stable w.r.t. $(1, 9, 30, I, 0.01)$, and the guaranteed cost value is $J^* = 0.7274 + 3.1858\|x_0\|^2$. Moreover, a stabilizing state-feedback gain matrix is given by

$$K = \begin{bmatrix} 0.0336 & -0.1319 & 0.3963 \\ 0.1686 & -1.8224 & 0.8318 \end{bmatrix}.$$

In the context of simulation outcomes, we have chosen the initial condition $x_0 = (0.7, 0.7, 0.7)^T \in \mathbb{R}^3$ and $\omega(t) = 0.1 \sin t$. Figure 4.1 in the thesis illustrates the response of $x^T(t)E^T REx(t)$ from the system (4.10) when there is no control input. It is conspicuous that the open-loop system does not meet the conditions for being FTB of Definition 4.1.3. Figure 4.2 in the thesis depicts the time evolution of $x^T(t)E^T REx(t)$ from the closed-loop system (4.11). From Figure 4.2 in the thesis, it is evident that the closed-loop system (4.11) is SFTB with respect to $(1, 9, 30, I, 0.01)$

Conclusions

The thesis investigates the qualitative properties of some singular differential systems with both integer-order and fractional-order derivatives. Specifically, it examines the exponential stability and stabilization of positive singular systems with delays, the finite-time boundedness of singular Caputo fractional differential systems with disturbances and uncertainties, as well as several related qualitative problems in control theory. The novel contributions of the thesis include:

- Establishing sufficient conditions to ensure the exponential stability and stabilization of discrete-time impulsive positive singular system with time delays.

- Establishing sufficient conditions for finite-time boundedness and designing a control function for the FTDC problem of Caputo fractional singular systems satisfying a one-sided Lipschitz condition.

- Establishing sufficient conditions for finite-time boundedness and designing a robust control function for the FTGCC problem of perturbed Caputo fractional singular systems.

Some future research directions

The following are the directions that we will continue to research after finishing the thesis.

- Study the stability and stabilization problems for the class of discrete-time positive singular fractional systems with delays.

- Investigate the stability problem in the sense of Lyapunov and several related control problems for classes of delayed singular fractional systems.

- Analyze the stability and finite-time boundedness problems, along with some related control problems, for classes of complex singular fractional systems.

List of published works related to the thesis

(CT1) N.H. Sau, M.V. Thuan, N.T. Phuong (2024), "Exponential stability for discrete-time impulsive positive singular system with time delays", *International Journal of Systems Science*, **55**(8), 1510–1527 (SCIE/Q1).

(CT2) N.T. Phuong, N.H. Sau, M.V. Thuan (2023), "Finite-time dissipative control design for one-sided Lipschitz nonlinear singular Caputo fractional order systems", *International Journal of Systems Science*, **54**(8), 1694–1712 (SCIE/Q1).

(CT3) N.T. Phuong, N.H. Sau, M.V. Thuan, N.H Muoi (2023), "Non- fragile finite-time guaranteed cost control for a class of singular Caputo fractional order systems with uncertainties", *Circuits, Systems, Signal Processing*, 43, 795–820 (SCIE/Q2).