THAI NGUYEN UNIVERSITY
Thai Nguyen University of Sciences

MAI THI NGOC HA

SOME ITERATIVE METHODS FOR SOLVING THE SPLIT FEASIBILITY PROBLEM WITH MULTIPLE OUTPUT SETS IN HILBERT SPACES

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Supervisors: Assoc. Prof. Dr. Truong Minh Tuyen
Assoc. Prof. Dr. Nguyen Thi Thu Thuy

First referee: ....
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Second referee: ....
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Third referee: ....
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Introduction

The split feasibility problem was first introduced by Censor and Elfving for modeling certain inverse problems in 1994. The split feasibility problem (SFP, for short) is formulated as follows:

$$\text{Find an element } x^\dagger \in C \text{ such that } T(x^\dagger) \in Q, \tag{SFP}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $T : H_1 \to H_2$ is a bounded linear operator (also known as a transfer operator).

It is easy to see that the convex feasibility problem is a special case of the (SFP). The (SFP) contains many applications to the image reconstruction problems, the intensity–modulated radiation therapy (IMRT) model, data transmission problems . . . by appropriately constructing the sets C, Q and the operator T.

To solve the (SFP), Censor and Elfving proposed the parallel iteration method and the cyclic iteration method based on the Bregman projection method in 1994. However, these algorithms have a drawback that each iteration involves computing inverse matrices. Calculating matrix inverses at each iterative step leads to high computational costs for large-scale practical problems. To overcome this limitation, in 2002, Byrne proposed the CQ algorithm in finite-dimensional spaces when the sets C and Q are chosen so that the orthogonal projections onto these sets can be easily computed.

Using an optimization approach, Xu developed the CQ algorithm to solve the (SFP) in infinite-dimensional real Hilbert spaces in 2010. The author showed that the iterative sequence generated by the CQ algorithm only converges weakly with the chosen step size depending on the norms of the transfer operator. The author also gave an example about the existence of the sets C, Q and the operator T in an infinite-dimensional real Hilbert space to show that the interative sequence generated by the CQ algorithm converges weakly but not strongly.

Xu showed that the solution set of the (SFP) coincides with the fixed point set of the nonexpansive mapping $S : H_1 \to H_1$ defined by

$$S := P_C\big[I - \gamma T^*(I - P_Q)T\big],$$

where $\gamma \in (0, 2/\|T\|^2)$, as well.

Therefore, the methods for finding fixed points of nonexpansive mappings can be applied to solve the (SFP) such as the Mann iteration method, the Halpern iteration method, the viscosity approximation method ...

A generalized form of the (SFP) is the multiple-set split feasibility problem (MSSFP, for short) which was proposed and studied by Censor et al. in 2005. Namely, let $C_1, C_2, \dots, C_N$ be N nonempty, closed, convex subsets of a real Hilbert space H_1 and $Q_1, Q_2, \dots, Q_M$ be M nonempty, closed, convex subsets of a real Hilbert space H_2. The (MSSFP) is formulated as follows:

$$\text{Find an element } x^\dagger \in \bigcap_{i=1}^{N} C_i \text{ such that } T(x^\dagger) \in \bigcap_{j=1}^{M} Q_j\ . \tag{MSSFP}$$

In 2006, Xu extended the CQ algorithm to solve the (MSSFP) based on approaching the fixed point method. The author proposed and proved the weak convergence of the iterative sequences defined by the Picard iteration algorithm, the parallel iteration algorithm and the cyclic iteration algorithm for solving the (MSSFP). To obtain the strong convergence, many mathematicians have studied the combination of the CQ method with the viscosity approximation method, the hybrid projection method, the shrinking projection method, the Halpern iteration method and others. However, these methods typically use a fixed step size that depends on information about the norm of the transfer operator T per each iteration. In general, calculating the operator norm T is not an easy task in practice. Therefore, establishing criteria for selecting the step size when the norm information of the transfer operator is unknown is a meaningful topic in computational practice. In recent years, many authors have studied to improve the CQ method for solving the (SFP) or the (MSSFP) so that the step size does not need any prior information of the norm of the transfer operators.

The (SFP) can be considered as a special case of the split common null point problem (SCNPP, for short). A generalized form of the (SCNPP) is the multiple-set split common null point problem (MSSCNPP, for short), which is stated as follows: Let $A_i : H_1 \to 2^{H_1}$, $i = 1, 2, \dots, N$ and $B_j : H_2 \to 2^{H_2}$, $j = 1, 2, \dots, M$ be maximal monotone operators in H_1 và H_2, respectively.

$$\text{Find an element} \quad x^\dagger \in \cap_{i=1}^{N} A_i^{-1}(0) \cap T^{-1}\big(\cap_{j=1}^{M} B_j^{-1}(0)\big). \tag{MSSCNPP}$$

Furthermore, we see that the (SCNPP) or the (MSCNPP) can be reduced to the split common fixed point problems (SCFPP). The generalized form of the (SCFPP, for short) is formulated as follows: Let $S_i : H_1 \to H_1$, $i = 1, 2, \dots, N$ and $\Xi_j :\ H_2 \to\ H_2$, $j = 1, 2, \dots, M$ be nonexpansive mappings on H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator.

$$\text{Find an element} \quad x^\dagger \in \cap_{i=1}^N \operatorname{Fix}(S_i) \cap T^{-1}\big(\cap_{j=1}^M \operatorname{Fix}(\Xi_j)\big). \qquad \text{(SCFPP)}$$

The (SFP), (SCFPP), (SCNPP) and the other related problems can be rewritten in the following generalized form: Let X and Y be two Hilbert or Banach spaces and let $T : X \to Y$ be a mapping from X into Y. Suppose that (P_1) and (P_2) are two given problems on X and Y, respectively. Consider the problem: Find an element $x^\dagger$ in X such that $x^\dagger$ is a solution of the Problem (P_1) and $T(x^\dagger)$ is a solution of the Problem (P_2). We denote this problem as (P). The generalized form of the (P) is stated as follows: Let $X_1, X_2, \ldots, X_N$ be Hilbert or Banach spaces and let $A_i : X_i \to X_{i+1}$, $i = 1, 2, \ldots, N-1$, be mappings from X_i into X_{i+1}. Suppose that (P_i), $i = 1, 2, \ldots, N$, are N given problems on X_i, respectively. Then the generalized form of the (P) is the problem of finding an element $x^\dagger$ on X_1 such that $x^\dagger$ is a solution to (P_1), $A_1(x^\dagger)$ is a solution to (P_2)… and $A_{N-1}(A_{N-2}(\ldots A_2(A_1(x^\dagger))))$ is a solution to (P_N). We denote this problem as (GP). There are many real-world problems that can be formulated in the form of the (GP). For example, the balance problem in a production line where the quantity of semi-finished products in the previous production process must be equal to the required quantity in the next production process. In 2019, Reich and Tuyen first proposed and studied the above problem model, which named the generalized split feasibility problem (GSFP, for short).

From the above analysis, we concern and study a several of class of the problems that are more general than the (GSFP).

• Firstly, we propose and study the split feasibility problem with multiple output sets (SFPMOS, for short). This problem is formulated as follows: Let H, H_i, $i = 1, 2, \ldots, N$ be real Hilbert spaces, $T_i : H \to H_i$, $i = 1, 2, \ldots, N$, be bounded linear operators, $C \subseteq H$ and $Q_i \subseteq H_i$, $i = 1, 2, \ldots, N$, be nonempty, closed and convex subsets.

$$\text{Find an element } x^\dagger \in \Omega^{\text{SFPMOS}} := C \cap (\cap_{i=1}^N T_i^{-1}(Q_i)) \neq \emptyset, \qquad \text{(SFPMOS)}$$

that is, $x^\dagger \in C$ and $T_i x^\dagger \in Q_i$ for all $i = 1, 2, \ldots, N$.

A practical example of the (SFPMOS) is the image classification problem via support vector machine learning with the MNIST dataset.

• Secondly, we concern and study the split common fixed point problem with multiple output sets (SCFPPMOS, for short). This problem is defined as follows: Let H and H_i be real Hilbert spaces and $T_i : H \to H_i$, $i = 1, 2, \ldots, N$ be bounded linear operators. Let $S_j : H \to H$, $j = 1, 2, \ldots, M$, $\Xi_k^i : H_i \to H_i$,

$i = 1, 2, \ldots, N,\ k = \ 1, 2, \ldots, M_i$ be nonexpansive mappings.

$$\text{Find an element } \ x^* \in \Omega^{\text{SCFPPMOS}}, \qquad \text{(SCFPPMOS)}$$

where $\Omega^{\text{SCFPPMOS}} := \big(\cap_{j=1}^{M} \text{Fix}(S_j)\big) \cap \Bigg(\cap_{i=1}^{N} T_i^{-1}\big(\cap_{k=1}^{M_i} \text{Fix}(\Xi_k^i)\big)\Bigg)$.

The objective of the thesis is to study and propose algorithms for solving classes of the split feasibility problems and the split common fixed point problems with multiple output sets in real Hilbert spaces. Specifically, the study objectives are as follows.

- Propose CQ-type algorithms combined with the Halpern iteration method and the viscosity approximation method to solve the (SFPMOS) and the (SCFPPMOS);
- Propose algorithms for solving the (SCFPPMOS) using the hybrid and shrinking projection techniques;
- Apply the proposed algorithms to some related problems;
- Provide and calculate some numerical examples to illustrate the effective proposed algorithms.

In addition to the introduction, conclusion and list of references, the thesis is presented in three chapters.

Chapter 1. Preliminaries

Chapter 2. Split feasibility problem with multiple output sets

Chapter 3. Split common fixed point problem with multiple output sets

Chapter 1

Preliminaries

This chapter presents some basic concepts and properties about real Hilbert spaces, convex sets and convex functions, metric projection, subdifferential and minimization of convex functions, nonexpansive mappings and several lemmas used in the next chapters.

1.1 A brief introduction to Hilbert spaces

1.2 Subdifferential and convex optimization problems

1.3 Nonexpansive mappings

1.4 Some auxilary lemmas

Chapter 2

Split feasibility problem with multiple output sets

In this chapter, we propose and study some interative methods to solve the split feasibility problem with multiple output sets in Hilbert spaces based on using optimal approaches. The content of this chapter is written based on the results of two articles (CT1) and (CT2) in the List of published works related to the thesis.

2.1 The optimization approachs to solving the split feasibility problem with multiple output sets

We first recall the split feasibility problem with multiple output sets mentioned in the Introduction: Let H, H_i, $i = 1, 2, \ldots, N$, be real Hilbert spaces and let $T_i : H \to H_i$, $i = 1, 2, \ldots, N$, be bounded linear operators. Let C and Q_i be nonempty, closed and convex subsets of H and H_i, $i = 1, 2, \ldots, N$, respectively.

$$\text{Find an element } x^\dagger \in C \text{ such that } T_i x^\dagger \in Q_i,\ \forall i = 1, 2, \ldots, N. \tag{2.1}$$

The solution set of Problem (2.1) is denoted by Ω^{SFPMOS}, that is

$$\Omega^{\text{SFPMOS}} = \{x^\dagger \in C \mid T_i x^\dagger \in Q_i,\ \forall i = 1, 2, \ldots, N\}.$$

In this chapter, we always suppose that $\Omega^{\text{SFPMOS}} \neq \emptyset$.

2.1.1 The first optimal approach

Let $g : H \to \mathbb{R}$ be a function defined by

$$g(x) := \frac{1}{2} \sum_{i=1}^{N} \|(I - P_{Q_i}) T_i x\|^2,\ \forall x \in H.$$

We see that g is a convex function on H. Besides, it is not hard to show that Problem (2.1) is equivalent to the following convex constrained optimization problem:

$$\min_{x\in C} g(x).$$

In other words, $x^\dagger$ is a solution to Problem (2.1) if and only if

$$0 \in \triangledown g(x^\dagger) + N_C(x^\dagger).$$

The above inclusion is equivalent to

$$0 \in \sum_{i=1}^{N} T_i^*(I - P_{Q_i})T_i x^\dagger + N_C(x^\dagger).$$

Using the definition of the normal cone to C at the point x and the properties of metric projection, we obtain

$$x^\dagger = P_C\big[x^\dagger - \gamma \sum_{i=1}^{N} T_i^*(I - P_{Q_i})T_i x^\dagger\big], \tag{2.2}$$

where γ is an arbitrary positive real number. It also follows from the equality (2.2) that an element $x^\dagger$ is a solution of Problem (2.1) if and only if it is a fixed point of the nonexpansive mapping:

$$P_C\big[I - \gamma \sum_{i=1}^{N} T_i^*(I - P_{Q_i})T_i\big].$$

From these analysis, we propose and study the following iterative algorithm for solving Problem (2.1).

Algorithm 2.1.1.

Step 0. – Choose $x_0 \in C$ arbitrary;

– Choose a sequence $\{\gamma_n\}$, which satifies

$$0 < a \leq \gamma_n \leq b < \frac{2}{N \max_{i=\overline{1,N}}\{\|T_i\|^2\}},\ n \geq 0. \tag{$\gamma1$}$$

Set $n := 0$.

Step 1. Compute

$$x_{n+1} = P_C\big[x_n - \gamma_n \sum_{i=1}^{N} T_i^*(I - P_{Q_i})T_i x_n\big]. \tag{2.3}$$

Step 2. Set $n := n + 1$ and go to **Step 1**.

We have the following result.

Theorem 2.1.1. *The sequence* $\{x_n\}$ *generated by Algorithm 2.1.1 converges weakly to a solution of Problem* (2.1).

Remark 2.1.2. (i) When $N = 1$ and $H = \mathbb{R}^n$, Algorithm 2.1.1 becomes the CQ Algorithm which is proposed by Byrne proposed in 2002 to solve Problem (SFP).

(ii) When $N = 1$, Algorithm 2.1.1 reduces to the algorithm proposed by Xu in 2010 to solve the (SFP) in an infinite-dimensional real Hilbert space.

In order to obtain the strong convergence, we combine Algorithm 2.1.1 with the Halpern iteration method. We get the following algorithm.

Algorithm 2.1.2.

Step 0. – Choose $x_0, u \in C$.
– Choose a sequence $\{\gamma_n\}$ satisfying the condition $(\gamma 1)$;
– Choose $\{\alpha_n\}$ satisfying condition

$$\{\alpha_n\} \subset (0,1),\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty. \tag{α}$$

Set $n := 0$.

Step 1. Compute

$$x_{n+1} = \alpha_n u + (1-\alpha_n) P_C\big[x_n - \gamma_n \sum_{i=1}^{N} T_i^*(I - P_{Q_i}) T_i x_n\big]. \tag{2.8}$$

Step 2. Set $n := n+1$ and go to **Step 1**.

The strong convergence of Algorithm 2.1.2 is given in the theorem below.

Theorem 2.1.3. *The sequence* $\{x_n\}$ *is generated by Algorithm 2.1.2 converges strongly to* $P_{\Omega^{\mathrm{SFPMOS}}}\, u$.

Next, we propose and study a more general algorithm when u is replaced by the value of a contraction mapping at a point x.

Algorithm 2.1.3.

Step 0. – Choose $y_0 \in C$;

– Choose a sequence $\{\gamma_n\}$ satisfying condition $(\gamma 1)$;

– Choose $\{\alpha_n\}$ satisfying condition (α);

– Choose $f : H \to C$ being a contraction mapping with a contraction coefficient $c \in [0, 1)$.

Set $n := 0$

Step 1. Compute y_{n+1} as follows:

$$y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) P_C[y_n - \gamma_n \sum_{i=1}^{N} T_i^*(I - P_{Q_i}) T_i y_n]. \tag{2.19}$$

Step 2. Set $n := n + 1$ and go to **Step 1**.

The strong convergence of the iterative sequence generated by Algorithm 2.1.3 is established in the following theorem.

Theorem 2.1.4. *The sequence $\{y_n\}$ is generated by Algorithm 2.1.3 converges strongly to a point $x^\dagger \in \Omega^{\mathrm{SFPMOS}}$, which is the unique solution to the variational inequality*

$$\langle (I - f)x^\dagger, y - x^\dagger \rangle \geq 0, \ \forall y \in \Omega^{\mathrm{SFPMOS}}. \tag{VIP($I - f, \Omega^{\mathrm{SFPMOS}}$)}$$

2.1.2 The second optimal approach

Now, we consider the function $h : H \to \mathbb{R}$ defined by

$$h(x) := (\max_{i=1,2,\ldots,N} f_i)(x), \ \forall x \in H$$

and

$$f_i(x) := \frac{1}{2} \|(I - P_{Q_i}) T_i x\|^2, \ i = 1, 2, \ldots, N.$$

It is not dificult to see that Problem (2.1) is equivalent to the following convex constrained optimization problem:

$$\min_{x \in C} h(x). \tag{2.21}$$

So, an element $x^\dagger$ is a solution of Problem (2.21) if and only if

$$0 \in \partial h(x^\dagger) + N_C(x^\dagger).$$

We have

$$\partial(\max_{i=1,2,\ldots,N} f_i)(x^\dagger) \supseteq \mathrm{co}\{ \bigcup_{i \in I(x^\dagger)} \partial f_i(x^\dagger) \},$$

where $I(x^\dagger) := \{i \in \{1, 2, \dots, N\} \mid f_i(x^\dagger) = (\max_{i=1,2,\dots,N} f_i)(x^\dagger)\}$. Thus, if the element $x^\dagger \in H$ satisfies

$$\mathrm{co}\{\bigcup_{i\in I(x^\dagger)} \partial f_i(x^\dagger)\} + N_C(x^\dagger)\} \ni 0 \tag{2.22}$$

then $x^\dagger$ is a solution of Problem (2.21) and it is also a solution of Problem (2.1).

We can prove that (2.22) is equivalent to

$$x^\dagger = P_C[x^\dagger - \gamma \sum_{i\in I(x^\dagger)} \lambda_i T_i^*(I - P_{Q_i})T_i x^\dagger], \tag{2.23}$$

where $\lambda_i \geq 0$ for all $i \in I(x^\dagger)$, $\sum_{i\in I(x^\dagger)} \lambda_i = 1$ and γ is an arbitrary positive real number.

The equality (2.23) suggests us to construct the two algorithms below that approximate to the solution to the Problem (2.1) based on the Halpern iteration method and the viscosity approximation method.

Algorithm 2.1.4.

Step 0. – Choose $x_0 \in C$ and $\{\rho_n\} \subset [a, b] \subset (0, 2)$ and set $n := 0$.

Step 1. Compute

$$x_{n+1} = P_C[x_n - \gamma_n \sum_{i\in I(x_n)} \lambda_{i,n} T_i^*(I - P_{Q_i})T_i x_n\}], \tag{2.24}$$

where $I(x_n) = \{i \mid \|T_i x_n - P_{Q_i} T_i x_n\| = \max_{i=1,2,\dots,N} \|T_i x_n - P_{Q_i} T_i x_n\|\}$, $\lambda_{i,n} \geq 0$ for all $i \in I(x_n)$, $\sum_{i\in I(x_n)} \lambda_{i,n} = 1$ and $d_n = \max_{i=1,2,\dots,N} \|T_i x_n - P_{Q_i} T_i x_n\|$, the step size $\{\gamma_n\}$ is defined by

$$\gamma_n = \begin{cases} \rho_n \dfrac{d_n^2}{\|\sum_{i\in I(x_n)} \lambda_{i,n} T_i^*(I - P_{Q_i})T_i x_n\|^2} & \text{if } \|\sum_{i\in I(x_n)} \lambda_{i,n} T_i^*(I - P_{Q_i})T_i x_n\| > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{$\gamma 2$}$$

Step 2. Set $n := n + 1$ and go to **Step 1**.

Theorem 2.1.5. *The sequence $\{x_n\}$ generated by Algorithm 2.1.4 converges weakly to an element in Ω^{SFPMOS}.*

The following corollary indicates that the above theorem still remains valid for a specific index $i_n \in I(x_n)$.

Corollary 2.1.6. *For any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by the following iterative method:*

$$\begin{cases} \text{Choose } i_n \text{ such that } \|T_{i_n}x_n - P_{Q_{i_n}}T_{i_n}x_n\| = \max_{i=1,2,\dots,N} \|T_ix_n - P_{Q_i}T_ix_n\|, \\ x_{n+1} = P_C[x_n - \gamma_n T_{i_n}^*(I - P_{Q_{i_n}})T_{i_n}x_n],\ n \geq 0, \end{cases}$$

where $\{\gamma_n\}$ is defined by

$$\gamma_n = \begin{cases} \rho_n \dfrac{\|T_{i_n}x_n - P_{Q_{i_n}}T_{i_n}x_n\|^2}{\|T_{i_n}^*(I - P_{Q_{i_n}})T_{i_n}x_n\|^2}, & \text{if } \|T_{i_n}^*(I - P_{Q_{i_n}})T_{i_n}x_n\| > 0, \\ 0, & \text{otherwise}, \end{cases}$$

and $\{\rho_n\} \subset [a,b] \subset (0,2)$. Then the sequence $\{x_n\}$ converges weakly to an element in Ω^{SFPMOS}.

Combining Algorithm 2.1.4 with the viscosity approximation method, we propose the following algorithm for solving Problem (2.1) and establish a strong convergence theorem for it.

Algorithm 2.1.5.

Step 0. – Choose $x_0 \in C$ arbitrary;
– Choose $\{\rho_n\} \subset [a,b] \subset (0,2)$;
– Choose $\{\alpha_n\}$ satisfying condition (α);
– Choose $f : H \to C$ being a contraction mapping with a contraction coefficient $c \in [0,1)$.
Set $n := 0$.

Step 1. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)P_C[x_n - \gamma_n \sum_{i \in I(x_n)} \lambda_{i,n} T_i^*(I - P_{Q_i})T_ix_n\}], \tag{2.33}$$

where $I(x_n) = \{i \mid \|T_ix_n - P_{Q_i}T_ix_n\| = \max_{i=1,\dots,N} \|T_ix_n - P_{Q_i}T_ix_n\|\}$, $\lambda_{i,n} \geq 0$ for all $i \in I(x_n)$, $\sum_{i \in I(x_n)} \lambda_{i,n} = 1$, the sequence $\{\gamma_n\}$ satisfies condition $(\gamma 2)$.

Step 2. Set $n := n+1$ and go to **Step 1**.

The following theorem presents the strong convergence of Algorithm 2.1.5.
Theorem 2.1.7. *The sequence $\{x_n\}$ generated by Algorithm 2.1.5 converges strongly to $x^\dagger$, which is the unique solution to the variational inequality* $(\text{VIP}(I - f, \Omega^{\text{SFPMOS}}))$.

The above theorem remains valid for specific $i_n \in I(x_n)$.

Corollary 2.1.8. *For any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by the following iterative method:*

$$\begin{cases} \text{Choose } i_n \text{such that } \|T_{i_n}x_n - P_{Q_{i_n}}T_{i_n}x_n\| = \max\limits_{i=1,2,\dots,N} \|T_i x_n - P_{Q_i}T_i x_n\|, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)P_C[x_n - \gamma_n T_{i_n}^*(I - P_{Q_{i_n}})T_{i_n}x_n], \ n \geq 0, \end{cases}$$

where $\{\gamma_n\}$ is defined by

$$\gamma_n = \begin{cases} \rho_n \dfrac{\|T_{i_n}x_n - P_{Q_{i_n}}T_{i_n}x_n\|^2}{\|T_{i_n}^*(I - P_{Q_{i_n}})T_{i_n}x_n\|^2}, & \text{if } \|T_{i_n}^*(I - P_{Q_{i_n}})T_{i_n}x_n\| > 0, \\ 0, & \text{otherwise}, \end{cases}$$

$\{\rho_n\} \subset [a,b] \subset (0,2)$ and $f : H \to C$ is a strict contraction mapping H into C with the contraction coefficient $c \in [0,1)$. If the sequence $\{\alpha_n\}$ satisfies the conditions (α) then the sequence $\{x_n\}$ converges strongly to $x^\dagger$, which is the unique solution to the variational inequality VIP$(I - f, \Omega^{\text{SFPMOS}})$.

2.2 Applications and numerical examples

2.2.1 An Application to the generalized split feasibility problem

We know that when $H = H_1$, $C = C_1$, $Q_i = C_{i+1}, 1 \leq i \leq N-1$, $T_1 = A_1$, $T_2 = A_2A_1, \dots$ and $T_{N-1} = A_{N-1}A_{N-2}\dots A_1$ then the (SFPMOS) becomes the (GSFP). So we can use the algorithms and the theorems in Section 2.1 to solve this problem. By substituting $T_1 = A_1$, $T_2 = A_2A_1, \dots$ and $T_{N-1} = A_{N-1}A_{N-2}\dots A_1$, the sequences generated by four proposed algorithms converge to a solution of the (GSFP).

2.2.2 Numerical examples

In this section, we present three numerical examples to illustrate the efficiency of the proposed algorithms. Example 2.2.5 uses Algorithms 2.1.1, 2.1.4, 2.1.3 and 2.1.5 to solve the generalized split feasibility problem in finite–dimensional Hilbert spaces and the sequence $\{x_n\}$ defined by Algorithms 2.1.3, 2.1.5 converges to exact solution $x^\dagger = (0,0,0,0)$ of the problem in this example. Example 2.2.6 uses Algorithms 2.1.3 and 2.1.5 to solve the split feasibility problem with multiple output sets in finite–dimensional Hilbert spaces and we see that the sequence $\{x_n\}$ generated by the above two algorithms converges to an element of the solution set of the problem under consideration and we cannot calculate this solution exactly. Finally, Example 2.2.7 illustrates the effectiveness of the proposed algorithms to solve the split feasibility problem with multiple output sets in infinite–dimensional Hilbert spaces. In this summary, we briefly present Example 2.2.7.

Example 2.2.7. In the Hilbert space $L^2[0,1]$, we take $a_0(t) = \sin t$ and $a_i(t) = t^i$ for all $i = \overline{1,100}$, and choose the following nonempty, closed and convex subsets:

$$C = \{x \in L^2[0,1] | \ \langle a_0, x\rangle \leq 1\}; \ Q_i = \{x \in L^2[0,1] | \ \langle a_i, x\rangle \leq 0\}, \ \forall i = \overline{1,100},$$

The linear bounded operators: $T_i : L^2[0,1] \to L^2[0,1]$ are given by $T_i x = ix$ for all $i = \overline{1,100}$.

We consider the following (SFPMOS):

$$\text{Find } x^* \in \Omega^{\text{SFPMOS}} := C \ \cap \left(\cap_{i=1}^{100} T_i^{-1}(Q_i) \right). \tag{2.44}$$

It is easy to see that $\Omega^{\text{SFPMOS}} \neq \emptyset$ because $x(t) = 0 \in \Omega^{\text{SFPMOS}}$.

We first examine the convergence of the iteractive sequence $\{x_n\}$ defined by Algorithm 2.1.1 with $\gamma_n = 10^{-6}$ and the convergence of the iteractive sequence $\{x_n\}$ generated by Algorithm 2.1.4 with γ_n which satisfies Theorem 2.1.5 where $\rho_n = 0.15$. We use the stopping run condition $\text{TOL}_n := \|x_{n+1} - x_n\| < \varepsilon$ where ε is a given tolerance.

To verify the convergence of the iteractive sequences generated by these algorithms to an element in Ω^{SFPMOS}, we further define the parameter.

$$m := \max\{\langle a_0, x_n\rangle - 1, \max_{i=\overline{1,100}} \{\langle a_i, T_i x_n\rangle\}.$$

Note that, if $m \leq 0$ then x_n is a solution to Problem (2.44).

With the initial point $x_0(t) = e^t$, we obtain the numerical results of Algorithms 2.1.1 and 2.1.4 which are shown in Table 2.4.

From the numerical results presented in Table 2.4, we see that Algorithm 2.1.4 converges faster than Algorithm 2.1.1 in this example.

Next, we consider the convergence of the sequence $\{x_n\}$ generated by Algorithm 2.1.3 and Algorithm 2.1.5 with $\alpha_n = n^{-0.5}$ and a contraction mapping $f : \ L^2[0,1] \to C$ is defined by $f(x) = P_C(0.25x)$, for all $x \in L^2[0,1]$, the parameter $\gamma_n = 5.10^{-5}$ for Algorithm 2.1.3 and γ_n which satisfies Theorem 2.1.5, where $\rho_n = 1.75$ for Algorithm 2.1.5. In this case, it is easy to see that $x^*(t) = 0$ is the unique solution to the variational inequality $\text{VIP}(I - f, \Omega^{\text{SFPMOS}})$.

Thus, we use the condition $\text{TOL}_n = \|x_n\| < \varepsilon$ to stop the iterative process, where ε is a given tolerance. The numerical results of Algorithms 2.1.3 and 2.1.5 are presented in Table 2.5

ε		**Algorithm 2.1.1**	**Algorithm 2.1.4**
10^{-4}	TOL_n	9.8880×10^{-5}	9.6610×10^{-5}
	n	138	58
	m	1.2567	0.0032
	Time (s)	0.1136	0.0362
10^{-5}	TOL_n	9.984×10^{-6}	9.9286×10^{-6}
	n	494	65
	m	1.0543	0.0010
	Time (s)	0.2629	0.0398
10^{-6}	TOL_n	9.9946×10^{-7}	7.3721×10^{-7}
	n	1717	73
	m	0.8486	2.7883×10^{-4}
	Time (s)	0.6776	0.0410
10^{-14}	TOL_n	1.0000×10^{-14}	9.1730×10^{-15}
	n	3808549	129
	m	0.0057	3.1103×10^{-8}
	Time (s)	1.2760×10^{3}	0.0621

Table 2.4: Table of numerical results for Algorithm 2.1.1 and 2.1.4

	Algorithm 2.1.3			**Algorithm 2.1.5**		
ε	TOL_n	n	Time (s)	TOL_n	n	Time (s)
10^{-4}	6.2940×10^{-5}	11	0.0137	8.0584×10^{-5}	10	0.0106
10^{-5}	6.6040×10^{-6}	16	0.0174	7.7238×10^{-6}	15	0.0127
10^{-6}	9.8876×10^{-7}	21	0.0309	7.6813×10^{-7}	21	0.0218

Table 2.5: Table of numerical results for Algorithm 2.1.3 and 2.1.5

Table 2.5 shows that the convergence rates of Algorithm 2.1.3 and Algorithm 2.1.5 are almost the same in this example.

Chapter 3

Split common fixed point problem with multiple output sets

In this chapter, we propose and study a number of iterative methods to approximate the solution to the split common fixed point problem with multiple output sets in real Hilbert spaces based on CQ-type algorithm and projection algorithms. The results of this chapter are written on the basis of articles (CT3) and (CT4) in the List of published works related to the thesis.

3.1 Algorithm and convergence

In this chapter, we consider the split common fixed point problem with multiple output sets mentioned in the introduction of the thesis.

3.1.1 A CQ-type algorithm

The proposed algorithm has the following scheme.

Algorithm 3.1.1.

Step 0. – Choose $x_0 \in H$ arbitrary;
– Choose $\{\rho_n\} \subset [c, d] \subset (0, 1)$;
– Choose $\{\alpha_n\}$ satisfying condition (α);
– Choose the sequence $\{a_n\}$ which is bounded ;
– Choose $f : H \to H$ which is a strict contraction mapping H into itself with the contraction coefficient $c \in [0, 1)$.
Set $n := 1$.

Step 1. Compute $y_{j,n} = S_j x_n$ for all $j = 1, 2, \ldots, M$ and let

$$d_n = \max_{j=1,2,\ldots,M} \{\|y_{j,n} - x_n\|\},$$
$$L_n = \{j \in \{1, 2, \ldots, M\} \mid \|y_{j,n} - x_n\| = d_n\}.$$

Step 2. Compute $z^i_{k,n} = \Xi^i_k(T_i x_n)$ for all $i = 1, 2, \dots, N$ and $k = 1, 2, \dots, M_i$, and let

$$d_{i,n} = \max_{k=1,2,\dots,M_i}\{\|z^i_{k,n} - T_i x_n\|\},\ i = 1, 2, \dots, N,$$
$$L_{i,n} = \{k \in \{1, 2, \dots, M_i\} \mid \|z^i_{k,n} - T_i x_n\| = d_{i,n}\}, i = 1, 2, \dots, N.$$

Step 3. Let $\Gamma_n := \max\big\{d_n, \max_{i=1,2,\dots,N}\{d_{i,n}\}\big\}$.

If $d_n = \Gamma_n$, then choose $j_n \in L_n$ and let $t_n = y_{j_n,n}$, $\Theta = I$.

Else, where $d_{i_n,n} = \Gamma_n$, choose $k_n \in L_{i_n,n}$, and let $t_n = z^{i_n}_{k_n,n}$, $\Theta = T_{i_n}$.

Step 4. Compute $u_n = x_n - \delta_n \Theta^*(\Theta x_n - t_n)$, where

$$\delta_n = \rho_n \frac{\|\Theta x_n - t_n\|^2}{\|\Theta^*(\Theta x_n - t_n)\|^2 + a_n}. \qquad (\delta_n)$$

Step 5. Compute $x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)u_n$, $n \geq 0$.

Set $n := n + 1$ and go to **Step 1**.

The following theorem yields the strong convergence of the sequence generated by Algorithm 3.1.1.

Theorem 3.1.3. *The sequence $\{x_n\}$ generated by Algorithm 3.1.1 converges strongly to an element $x^\dagger \in \Omega^{\text{SCFPPMOS}}$, which is the unique solution of the variational inequality* (VIP$(I - f, \Omega^{\text{SFPMOS}})$) *where Ω^{SFPMOS} replaced by Ω^{SCFPPMOS}.*

3.1.2 Hybrid projection algorithm

Algorithm 3.1.2. For any starting point $x_0 \in H$ and set $n := 1$, the scheme of the hybrid projection algorithm consists of five steps with Steps 1, 2, 3 implemented as Algorithm 3.1.1 and Steps 4, 5 are performed as follows:

Step 4. Define the subsets C_n and Q_n of H as follows:

$$C_n = \{z \in H \mid \|t_n - \Theta z\| \leq \|\Theta x_n - \Theta z\|\},$$
$$Q_n = \{z \in H \mid \langle x_0 - x_n, z - x_n\rangle \leq 0\}.$$

Step 5. Compute $x_{n+1} = P_{C_n \cap Q_n} x_0$. Set $n := n + 1$ and go to **Step 1**.

The strong convergence of the sequence $\{x_n\}$ defined by Algorithm 3.1.2 is demonstrated in the following theorem.

Theorem 3.1.7. *Suppose the assumptions of the* (SCFPPMOS) *are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 3.1.2 converges strongly to $x^\dagger = P_{\Omega^{\text{SCFPPMOS}}}\, x_0$.*

3.1.3 Shrinking projection algorithm

Algorithm 3.1.3. For any starting point $x_0 \in H$, let $C_0 = H$ and $n := 1$, the scheme of the shrinking projection algorithm consists of five steps with Steps 1, 2, 3, also implemented as Algorithm 3.1.1 and Steps 4, 5 are performed as follows:

Step 4. Define the subset C_{n+1} of H as follows:

$$C_{n+1} = \{z \in C_n \mid \|t_n - \Theta z\| \leq \|\Theta x_n - \Theta z\|\}.$$

Step 5. Compute $x_{n+1} = P_{C_{n+1}} x_0$. Set $n := n + 1$ and go to **Step 1**.

Theorem 3.1.12. *Suppose the assumptions of the* (SCFPPMOS) *are satisfied. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.1.3 converges strongly to* $x^\dagger = P_{\Omega^{\text{SCFPPMOS}}} x_0$.

3.2 Applications

3.2.1 The split feasibility problem with multiple output sets

We consider the split feasibility problem with output sets in a more general case than the problem stated in the Introduction. The problem is stated as follows: Let H, H_i, $i = 1, 2, \ldots, N$ be the real Hilbert spaces . Let $T_i : H \to H_i$, $i = 1, 2, \ldots, N$ be bounded linear operators. Let C_j, $j = 1, 2, \ldots, M$ be closed and convex subsets of H. Let Q^i_k, $i = 1, 2, \ldots, N$ where $k = 1, 2, \ldots, M_i$, be closed and convex subsets of H_i, .

$$\begin{aligned}&\text{Find an element } x^\dagger \in C_i,\ \forall j = 1, 2, \ldots, M \text{ such that}\\ &T_i x^\dagger \in Q^i_k,\ \forall i = 1, 2, \ldots, N;\ k = 1, 2, \ldots, M_i.\end{aligned} \tag{GSFPMOS}$$

The solution set of the (GSFPMOS) is denoted by Ω^{GSFPMOS} and we always suppose $\Omega^{\text{GSFPMOS}} \neq \emptyset$.

In Algorithm 3.1.1, 3.1.2 and 3.1.3, replacing $S_j = P_{C_j}$ for all $j = 1, 2, \ldots, M$ and $\Xi^i_k = P_{Q^i_k}$ for all $i = 1, 2, \ldots, N$, $k = 1, 2, \ldots, M_i$, we obtain CQ-type algorithm, hybrid projection algorithm and shinking projection algorithm to solve the (GSFPMOS) without any information about the norm of the transfer operator.

3.2.2 The split common fixed point problem for nonexpansive mappings

Applying Algorithm 3.1.1, 3.1.2 and 3.1.3 in the case of $N = 1$, we get three algorithms to solve the split common fixed point problem for nonexpansive mappings.

3.2.3 The fixed point problem for nonexpansive mappings

Applying Algorithm 3.1.1, 3.1.2 and 3.1.3, we also have corresponding algorithms to solve the fixed point problem for non-expansive mappings presented in Corollaries 3.2.5, 3.2.6 and 3.2.7 of the thesis.

3.3 Numerical examples

In this section, we introduce three numerical examples to illustrate the effectiveness of the proposed algorithms. Examples 3.3.1 and 3.3.2 consider the (SFPMOS) and the (SCFPPMOS) in finite–dimensional Hilbert spaces. Example 3.3.1 illustrates that the iterative sequence generated by the proposed algorithms converges to an element in the solution set of the problem, but we can not accurately determine this solution. Example 3.3.2 shows that the iterative sequence generated by the proposed algorithms converges to an exact solution of this problem. Example 3.3.3 solves the problem in Example 2.2.7 using the hybrid projection method and the shinking projection method. In this summary, we briefly present Example 3.3.2.

Example 3.3.2. Let g, g_1, g_2, g_3 and g_4 be the functions on $\mathbb{R}^5$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^6$, respectively, which are defined as follows:

$$
\begin{aligned}
g(x) &= \frac{1}{2}(x_1 - x_2 + x_3 - x_4 - 2x_5 - 1)^2, \text{ for all } x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5;\\
g_1(y) &= \frac{1}{2}(y_1 + y_2 - 5)^2, \text{ for all } y = (y_1, y_2) \in \mathbb{R}^2;\\
g_2(z) &= \frac{1}{2}(2z_1 + z_2 - z_3 - 4)^2, \text{ for all } z = (z_1, z_2, z_3) \in \mathbb{R}^3;\\
g_3(u) &= \frac{1}{2}(u_1 - u_2 - u_3 + u_4 - 1)^2, \text{ for all } u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4;\\
g_4(v) &= \frac{1}{2}(v_1 + 2v_2 - v_3 + v_4 + v_5 + v_6)^2, \text{ for all } v = (v_1, v_2, v_3, v_4, v_5, v_6) \in \mathbb{R}^6.
\end{aligned}
$$

The transfer mappings

$$
\begin{aligned}
&T_1 : \mathbb{R}^5 \to \mathbb{R}^2 \text{ is defined by } T_1 y = [y_1, y_2]^\top;\\
&T_2 : \mathbb{R}^5 \to \mathbb{R}^3 \text{ is defined by } T_2 z = [z_1, z_2, z_3]^\top;\\
&T_3 : \mathbb{R}^5 \to \mathbb{R}^4 \text{ is defined by } T_3 u = [u_1, u_2, u_3, u_4]^\top;\\
&T_4 : \mathbb{R}^5 \to \mathbb{R}^6 \text{ is defined by } T_4 v = [v_1, v_2, v_3, v_4, v_5, v_6]^\top.
\end{aligned}
$$

The representing matrices of the transfer mappings T_1, T_2, T_3 and T_4 are, respectively,

$$T_1 = \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 2 & -2 & 1 & -4 & -4 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 2 \\ 1 & 3 & -4 & 3 & 6 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 2 & 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & 2 & 3 \end{pmatrix}, T_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & -2 & -2 & -10 \end{pmatrix}.$$

We consider the following problem: Find an element $x^* \in \mathbb{R}^5$ such that

$$x^* \in \arg\min_{x \in \mathbb{R}^5} g(x); \quad T_1 x^* \in \arg\min_{y \in \mathbb{R}^2} g_1(y); \quad T_2 x^* \in \arg\min_{z \in \mathbb{R}^3} g_2(z);$$
$$T_3 x^* \in \arg\min_{u \in \mathbb{R}^4} g_3(u); \quad T_4 x^* \in \arg\min_{v \in \mathbb{R}^6} g_4(v) \ .$$

Let Ω be a solution set of this problem. It is not difficult to check that g and g_i, $i = 1, 2, 3, 4$ are convex functions and $\Omega = \{(a-b+c+3, a, b, c, 1) : \ a, b, c \in \mathbb{R}\}$.

We can see that this problem is equivalent to the problem: Find an element $x^* \in \mathbb{R}^5$ such that

$$x^* \in \nabla g^{-1}(0); T_1 x^* \in \nabla g_1^{-1}(0); T_2 x^* \in \nabla g_2^{-1}(0);$$
$$T_3 x^* \in \nabla g_3^{-1}(0); T_4 x^* \in \nabla g_4^{-1}(0).$$

In this example, ∇g and ∇g_i, $i = 1, 2, 3, 4$ are maximal monotone operators and, thus, $(I + \nabla g)^{-1}$ and $(I + \nabla g_i)^{-1}$ for all $i = 1, 2, 3, 4$ are nonexpansive maps. Therefore, the above problem is equivalent to finding the split common fixed point problem with multiple output sets: Find an element $x^* \in \mathbb{R}^5$ such that

$$x^* \in \mathrm{Fix}\left((I + \nabla g)^{-1}\right); T_1 x^* \in \mathrm{Fix}\left((I + \nabla g_1)^{-1}\right); T_2 x^* \in \mathrm{Fix}\left((I + \nabla g_2)^{-1}\right);$$
$$T_3 x^* \in \mathrm{Fix}\left((I + \nabla g_3)^{-1}\right); \quad T_4 x^* \in \mathrm{Fix}\left((I + \nabla g_4)^{-1}\right).$$

We denote Ω^{SCFPPMOS} as a solution set of the Problem (SCFPPMOS). We have $\Omega^{\text{ SCFPPMOS}} \equiv \Omega = \{(a - b + c + 3, a, b, c, 1) : \ a, b, c \in \mathbb{R}\}$.

We now test the convergence of Algorithms 3.1.1, 3.1.2 and 3.1.3, where the nonexpansive mappings $S_1 = (I + \nabla g)^{-1}; \Xi_i = (I + \nabla g_i)^{-1}$ for all $i = 1, 2, 3, 4$, the parameters $\alpha_n = n^{-1}, a_n = 0.00001, \rho_n = 0.95, x_0 = (1, -1, 1, -1, 1)$, the contraction mapping $f : \mathbb{R}^5 \to \mathbb{R}^5$ is defined by $f(x_n) = x_0$, for all $n \geq 1$. The sequence $\{x_n\}$ gernerated by the above algorithms converges strongly to $x^* = P_{\Omega^{\text{SCFPPMOS}}}\, x_0 = (0.75, -0.75, 0.75, -0.75, 1)$. In this case, we can easily see that x^* is also the unique solution to the variational inequality $\text{VIP}(I - f, \Omega^{\text{SFPMOS}})$ in which we replace Ω^{SFPMOS} by Ω^{SCFPPMOS}. Thus, we use the stopping run condition $\text{TOL}_n := \|x_n - x^*\| < \varepsilon$, where ε is a given tolerance.

With the starting point $x_0 = (1, -1, 1, -1, 1)$, Table 3.3 illustrates the convergence of the three proposed algorithms.

ε		**Algorithm 3.1.1**	**Algorithm 3.1.2**	**Algorithm 3.1.3**
10^{-4}	TOL_n	9.9881×10^{-5}	9.9141×10^{-5}	5.2862×10^{-5}
	n	621	175	14
	Time (s)	0.0400	1.3368	0.0820
10^{-5}	TOL_n	9.9965×10^{-6}	9.9752×10^{-6}	4.6409×10^{-6}
	n	2526	979	19
	Time (s))	0.1313	5.9919	0.1082
10^{-6}	TOL_n	9.9998×10^{-7}	9.9967×10^{-7}	9.1672×10^{-7}
	n	23854	3899	23
	Time (s)	1.1318	26.6064	0.1204

Table 3.3: Table of numerical results for Algorithm 3.1.1; 3.1.2 and 3.1.3

The numerical results in Table 3.3 show that, for the same given tolerance, Algorithm 3.1.3 requires the least amount of time and number of iterations, while Algorithm 3.1.2 requires fewer iterations than Algorithm 3.1.1 but takes more time.

Conclusions

The thesis focuses on proposing and studying the algorithms for solving the (SFPMOS) and the (SCFPPMOS). New algorithms are studied based on the CQ method, the viscosity approximation method, the Halpern iteration method and projection method.

The thesis has achieved the following results

- We have proposed five algorithms to solve the (SFPMOS) (Algorithm 2.1.1–2.1.5). We have also proposed and proven the weak and the strong convergence theorems for these algorithms. We developed these algorithms from Byrne's CQ algorithm for the split feasibility problem in finite–dimensional Hilbert spaces based on the optimal approach. The advantage of Algorithms 2.1.1–2.1.3 is the simplicity of calculation at each iteration. The advantage of Algorithms 2.1.4 and 2.1.5, besides the simplicity of computation at each iteration, is that the step size does not depend on any prior information about the norms of the tranfer operators. These results are shown in the works (CT1) and (CT2).
- We have proposed three algorithms to approximate the solution of the (SCFPPMOS) (Algorithm 3.1.1–3.1.3). We have proposed and proven the weak and the strong convergence theorems for these algorithms, too. By using CQ method, hybrid projection techniques and shinking projection techniques combined with the viscosity approximation method, we have designed algorithms with self-adaptive step sizes. These results are presented in the works (CT3) and (CT4).
- These algorithms are applied to the generalized split feasibility problem, the split feasibility problems with multiple output sets, the split common fixed point problem for nonexpansive mappings or the fixed point problem for nonexpansive mappings. Numerical experimental results in finite–dimensional real Hilbert space as well as infinite– dimensional real Hilbert space have shown the effectiveness of the proposed methods.

Some future study directions

The following are the directions that we will continue to study after finishing the thesis.

- Study the split feasibility problem and the related problems in Banach spaces.
- Explore the (SFPMOS) when at least one constraints set is not convex or at least one transfer operator is nonlinear.
- Study the stability of algorithms when the input data is noisy.
- Evaluate the convergence speed of algorithms.

List of published works related to the thesis

(CT1) Reich S., Tuyen T.M., Ha M.T.N. (2020), “The split feasibility problem with multiple output sets in Hilbert spaces”, *Optim. Lett.*, **14**, pp. 2335–2353 (SCIE-Q2).

(CT2) Reich S., Tuyen T.M., Ha M.T.N. (2021), “An optimization approach to solving the split feasibility problem in Hilbert spaces”, *J. Global Optim.*, **70**, pp. 837–852 (SCIE-Q1).

(CT3) Kim J.K., Tuyen T.M., Ha M.T.N. (2021), “Two projection methods for solving the split common fixed point problem with multiple output sets in Hilbert spaces”. *Numer. Funct. Anal. Optim.* **42**(8), pp. 973–988 (SCIE-Q2).

(CT4) Reich S., Tuyen T.M., Thuy N.T.T., Ha M.T.N. (2022). “A new self-adaptive algorithm for solving the split common fixed point problem with multiple output sets in Hilbert spaces”, *Numer. Algorithms*, **89**, pp. 1031–1047 (SCIE-Q1).